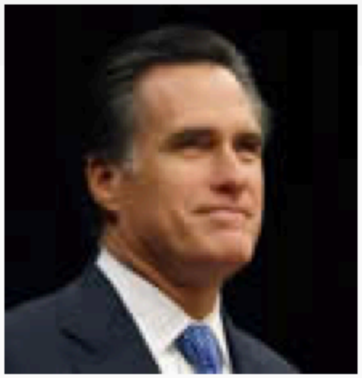
# Efficient Market Making Via Optimization & Connections to Online Learning

Jake Abernethy, Yiling Chen, and Jenn Wortman Vaughan

(Jenn's slides for CS269, Winter 2012)

# Information Markets



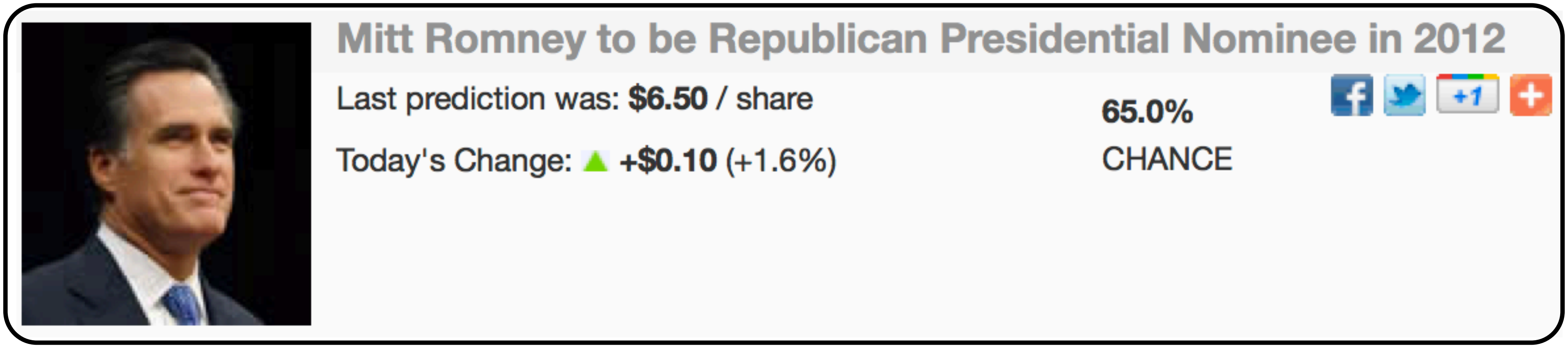
**Mitt Romney to be Republican Presidential Nominee in 2012**

Last prediction was: **$6.50** / share

Today's Change: ▲ **+$0.10** (+1.6%)

**65.0%**
CHANCE

# Information Markets




Potential payoff is \$10. If I think that the probability of this event is $p$, I should

- Buy this security at any price less than \$10$p$
- Sell this security at any price greater than \$10$p$

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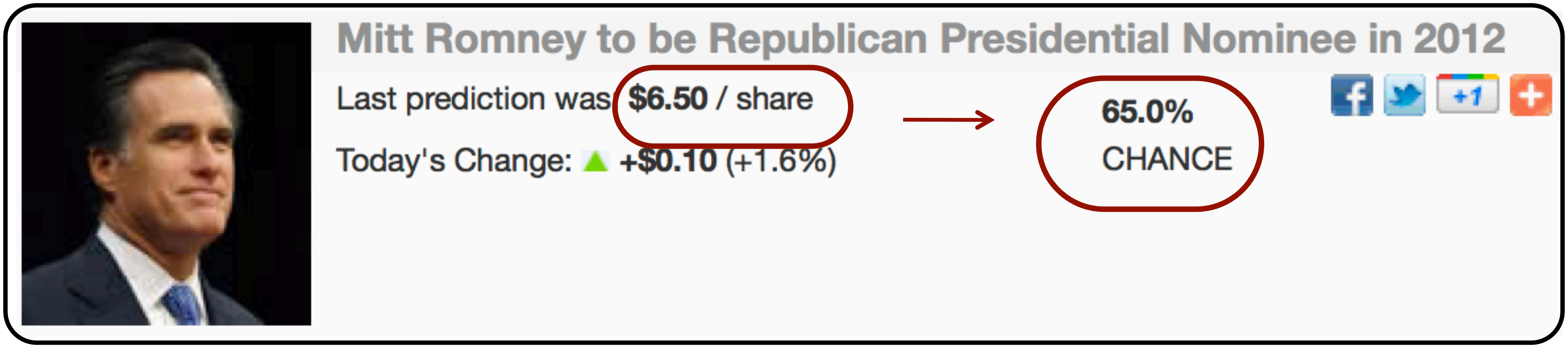


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Current price measures the population's collective beliefs

# In case you're curious…

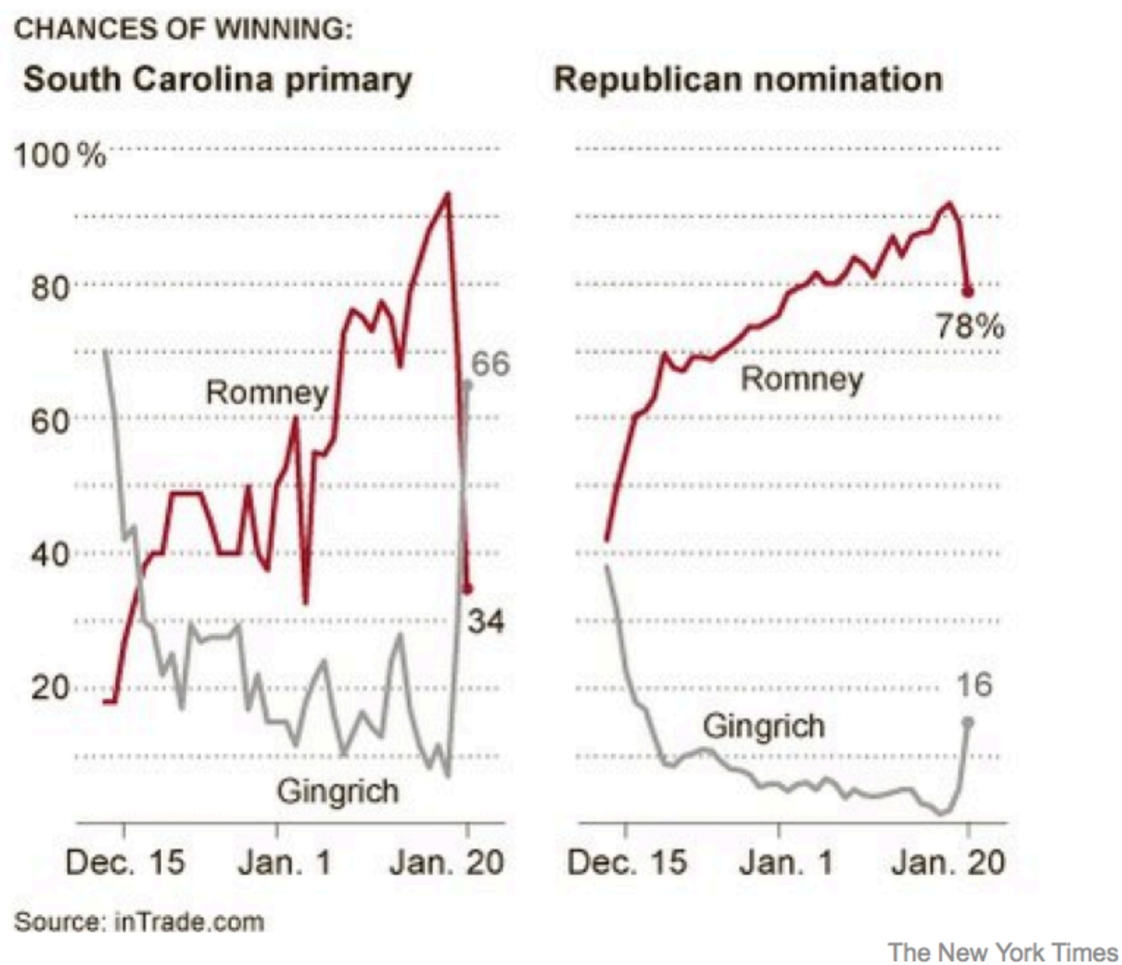
# The Caucus

The Politics and Government Blog of The Times


January 20, 2012, 8:21 PM

## Gingrich Gets a Boost on Intrade

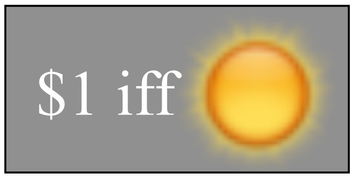
By RITCHIE S. KING




Newt Gingrich's presidential candidacy is on the upswing in South Carolina, if Intrade, a popular online market for political bets, is any indication.

# Market Makers for Complete Markets

- In a complete market, a security is offered for each potential state of the world



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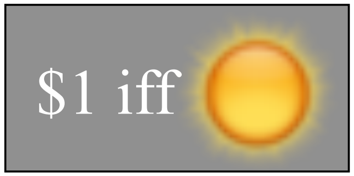


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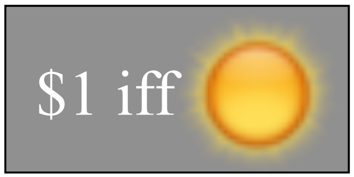


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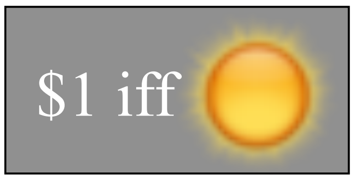


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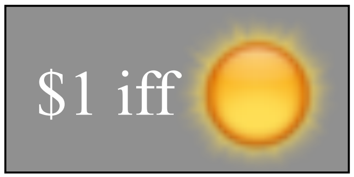




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# Example: LMSR

The Logarithmic Market Scoring Rule [Hanson, 2003] uses an exponential cost function

$$C(q_1,\ldots,q_N) = b\log \sum_{i=1}^{N} \exp(q_i/b)$$

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Notice that $p_i$ is increasing in $q_i$ and the prices sum to 1

# Complex Outcome Spaces


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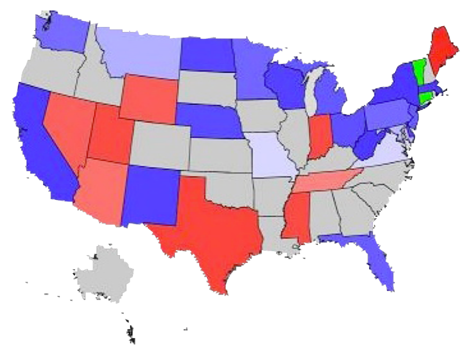

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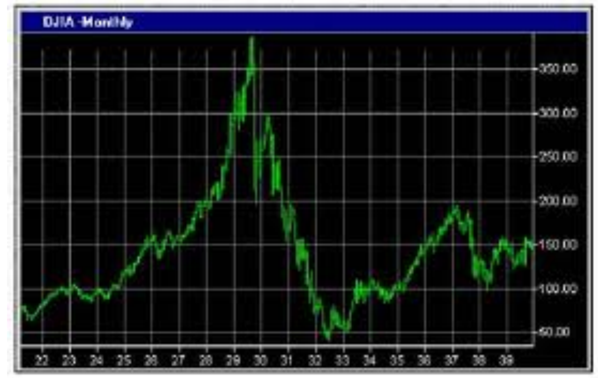


Infinite

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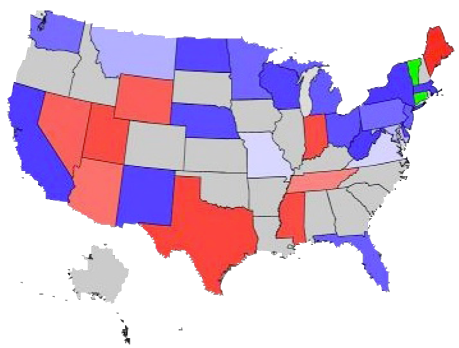

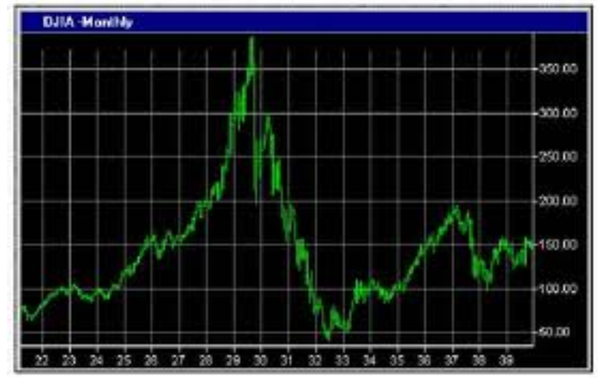


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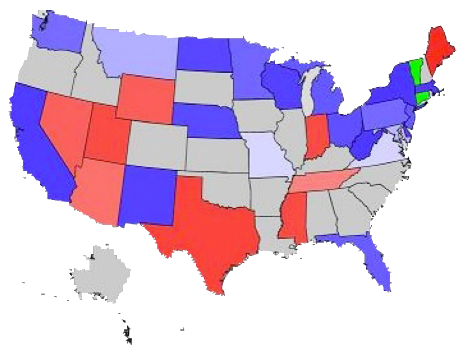

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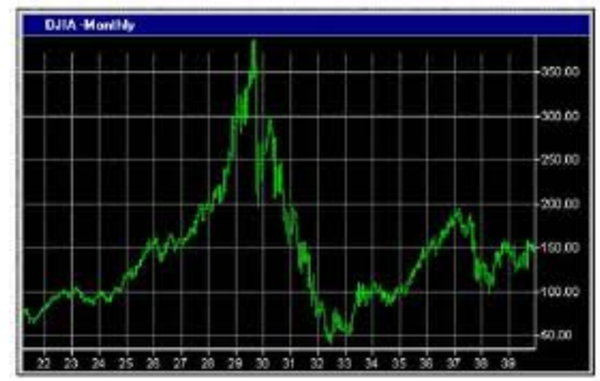


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- Cannot simply run a standard market like LMSR
  - Calculating prices is intractable
  - Reasoning about probabilities is too hard for traders
- Can run separate, independent markets (e.g., horses to win, place, or show) but this ignores logical dependences

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**Our Goal:** Given a small set of securities over a very large (or infinite) state space, design a consistent market that can be operated efficiently

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**Key Tools:** Convex optimization and conjugate duality

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*securities*

*states*

| | | | | | |
|---|---|---|---|---|---|
| 10 | 0 | 5.5 | 0 | 17 | 0 |
| .9 | .9 | .9 | 0 | .9 | .9 |
| 0 | 42 | 0 | 10 | 10 | 10 |
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| | A<B | B<A | A<C | C<A | B<C | C<B |
|---|---|---|---|---|---|---|
| ABC | 1 | 0 | 1 | 0 | 1 | 0 |
| ACB | 1 | 0 | 1 | 0 | 0 | 1 |
| BAC | 0 | 1 | 1 | 0 | 1 | 0 |
| BCA | 0 | 1 | 0 | 1 | 1 | 0 |
| CAB | 1 | 0 | 0 | 1 | 0 | 1 |
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In general… ???

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This alone implies the existence of a cost potential function!

$$\begin{aligned}&\text{Cost}(\mathbf{r} \mid \mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_t})\\ &\quad = C(\mathbf{r_1} + \mathbf{r_2} + \ldots + \mathbf{r_t} + \mathbf{r}) - C(\mathbf{r_1} + \mathbf{r_2} + \ldots + \mathbf{r_t})\end{aligned}$$

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- **Information incorporation:** The purchase of a bundle **r** should never cause the price of **r** to decrease
- **No arbitrage:** It is never possible to purchase a bundle **r** with a guaranteed positive profit regardless of outcome
- **Expressiveness:** A trader must always be able to set the market prices to reflect his beliefs

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**Theorem:** Under these five conditions, costs must be determined by a convex cost function $C$ such that

$$\{\nabla C(\mathbf{q}) : \mathbf{q} \in \mathrm{R}^K\} = \mathrm{Hull}(\boldsymbol{\rho})$$

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To generate a convex cost function $C$, we just have to choose an appropriate conjugate function and domain!

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We can borrow ideas from online linear optimization, and in particular, Follow the Regularized Leader algorithms

- Our conjugate function ≈ their regularizer

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Gives us a way to optimize trade-offs in market design!

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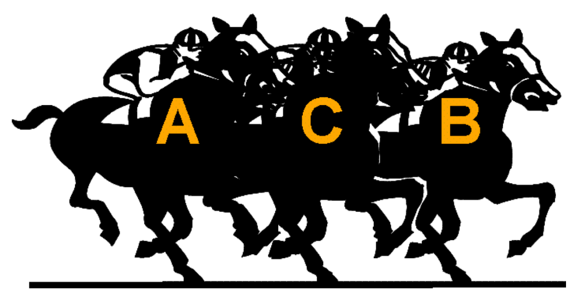

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- Both known to be #P-hard to price using LMSR [Chen et al., 2008]

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- Suppose our state space is all permutations of $n$ items (e.g., candidates in an election, or horses in a race)
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  - Subset bets: Bets on events of the form "horse $i$ finishes in position $j$" for any $i, j$


- Both known to be #P-hard to price using LMSR [Chen et al., 2008]
- Our framework handles both!

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- Easily handled by our framework!

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  - But does it increase worst case loss? No!

# Summary

- Our new optimization-based framework allows for the design of efficient market maker mechanisms for combinatorial or infinite state spaces

- Properties like worst-case loss and speed of price changes can be inferred easily

- Using this framework, we can design efficient markets for betting languages that are intractable to price using LMSR