Removing Arbitrage from Wagering Mechanisms

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We observe that Lambert et al.'s [2008] family of weighted score wagering mechanisms admit arbitrage: participants can extract a guaranteed positive payoff by betting on any prediction within a certain range. In essence, participants leave free money on the table when they "agree to disagree," and as a result, rewards don't necessarily go to the most informed and accurate participants. This observation suggests that when participants have immutable beliefs, it may be possible to design alternative mechanisms in which the center can make a profit by removing this arbitrage opportunity without sacrificing incentive properties such as individual rationality, incentive compatibility, and sybilproofness. We introduce a new family of wagering mechanisms called *no-arbitrage wagering mechanisms* that retain many of the positive properties of weighted score wagering mechanisms, but with the arbitrage opportunity removed. We show several structural results about the class of mechanisms that satisfy no-arbitrage in conjunction with other properties, and provide examples of no-arbitrage wagering mechanisms with interesting properties.

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1. INTRODUCTION

Betting markets of various forms, including stock exchanges [Grossman 1976], futures and options markets [Roll 1984], sports betting markets [Gandar et al. 1999], racetrack pari-mutuel systems [Thaler and Ziemba 1988; Plott et al. 1997], and modern prediction markets [Forsythe and Lundholm 1990; Forsythe et al. 1991; Berg et al. 2001; Wolfers and Zitzewitz 2004], have demonstrated their ability to incentivize participants to reveal their information about underlying events. Market prices have a history of equaling or beating other forecasts of events in domains from politics to product launches.

However, betting markets can induce complicated strategic play and obfuscate individual-level information. While the dynamics of betting markets may facilitate information aggregation [Ostrovsky 2012], it has been shown, both theoretically [Allen and Gale 1992; Kumar and Seppi 1992; Chakraborty and Yilmaz 2004; Chen et al. 2010; Gao et al. 2013] and empirically [Hansen et al. 2001], that market participants may misrepresent their private information to mislead other participants and profit

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later. If the price is too high, some traders will opt out, revealing nothing. Others may not reveal their full information due to budget constraints.

In many settings, a one-shot interaction is preferable to a continuous market. Participants have a simple, truthful dominant strategy and the center does not need to wait for equilibrium. The center obtains all participants' private beliefs and can then post-process the beliefs in one or more ways. For example, the average or weighted average of private beliefs can provide a robust prediction, with accuracy improving in the number and diversity of individual beliefs [Jacobs 1995; Surowiecki 2004; Chen et al. 2005; Dani et al. 2006; Page 2007; Ungar et al. 2012]. Researchers have developed one-shot betting mechanisms with various theoretical properties [Kilgour and Gerchak 2004; Johnstone 2007; Lambert et al. 2008, 2014]. In particular, Lambert et al. [2008] proposed a class of *weighted score wagering mechanisms* (WSWMs), where each participant makes a prediction about an uncertain event and wagers some amount of money. The total amount wagered is redistributed among the participants after the event outcome is revealed. These mechanisms satisfy a set of desirable properties, including (1) *budget balance*: the sum of participants' payoffs is zero and the center does not need to subsidize the betting; (2) *individual rationality*: each participant prefers participating to not participating; (3) *incentive compatibility*: each participant maximizes his expected payoff by predicting his true belief, and (4) *sybilproofness*: no participant can benefit from splitting his wager and participating under multiple identities. Conversely, Lambert et al. [2008] showed that any one-shot wagering mechanism satisfying this set of properties must be a WSWM.

Kilgour and Gerchak [2004], Johnstone [2007], and Lambert et al. [2008, 2014] designed their mechanisms for participants with immutable beliefs. While Bayesian agents sharing a common prior can never "agree to disagree", immutable agents persist in their disagreement, never wavering from their prior beliefs even upon observing other players' actions. Both extremes are abstractions, but the latter matches reality more closely—disagreement is rampant on trading and wagering platforms of all types. The immutable beliefs model does not suffer from the perverse conclusions of the no-trade theorem [Milgrom and Stokey 1982] that rational risk-neutral participants won't engage in any zero-sum wager. As a concrete example of a scenario in which the assumption of immutable beliefs is reasonable, consider an online ad exchange that engages companies like BlueKai, who possess a large amount of behavioral data, to bet on an arriving user's conversion rate for each of several competing ads. In order to inform ad placement, betting must happen in milliseconds and companies will use machine learning algorithms to form their predictions. In this setting, one-shot mechanisms seem all but necessary—a continuous market would be too slow for the exchange and too complex for strategic automated traders.

In this paper, we continue focusing on one-shot betting mechanisms for participants with immutable beliefs. We show that, in the WSWMs, participants leave free money on the table whenever they disagree. In other words, *there exist arbitrage opportunities in the WSWMs* — a participant can obtain a guaranteed positive payoff by betting on any prediction within certain range. This observation suggests that when participants have immutable beliefs, the center may be able to design mechanisms to make a profit by exploiting the arbitrage opportunities without sacrificing incentive properties such as individual rationality, incentive compatibility, and sybilproofness. This paper designs and analyzes such arbitrage-free wagering mechanisms.

Our main contributions are the following:

(1) We characterize the arbitrage opportunities in the WSWMs and define the *arbitrage interval* (Theorem 3.3).
(2) We show that this arbitrage property is not unique to the WSWMs and provide a sufficient condition for it to exist (Theorem 3.4).

(3) We propose a new class of mechanisms called *no-arbitrage wagering mechanisms* (NAWMs) (Definition 4.2) that are arbitrage-free, individually rational, incentive compatible, and anonymous, and give sufficient conditions for these mechanisms to be weakly budget balanced (i.e., the center always breaks even or profits, Theorem 4.4) and neutral (i.e., it is invariant to relabeling outcomes, Theorem 4.6).
(4) For a subclass of NAWMs with payoff functions of a particular functional form (Definition 5.1), we characterize the necessary and sufficient conditions for these mechanisms to be weakly budget balanced and neutral (Theorem 5.3), and prove that these mechanisms are sybilproof (Theorem 5.8).
(5) We provide specific examples of such mechanisms with certain interesting properties. For instance, we give a mechanism that makes the same profit for all outcomes, which can be interpreted as spreading the profit equally over all outcomes or maximizing the minimum profit. We give another mechanism that has the property that if everyone predicts that outcome 0 is more likely than outcome 1, and outcome 0 occurs, then the mechanism is exactly budget balanced (Theorem 5.6). In some sense, the center doesn't make a profit when all agents "correctly" predict the outcome.

Due to lack of space, some proofs are omitted. They appear in the full version of the paper available on the authors' websites.

1.1. Other Related Work

Proper scoring rules have been widely studied for one-shot information elicitation from individuals [Brier 1950; Good 1952; Winkler 1969; Savage 1971; Matheson and Winkler 1976; Gneiting and Raftery 2007]. Using a proper scoring rule, the center interacts with each individual independently and in general needs to pay the individuals for their predictions. Proper scoring rules are the building blocks for the budget-balanced mechanisms of Kilgour and Gerchak [2004], Johnstone [2007], and Lambert et al. [2008, 2014], as well as for our no-arbitrage wagering mechanisms in this paper. We discuss some basics of proper scoring rules in Section 2.3.

In addition to the WSWMs, the *competitive scoring rules* [Kilgour and Gerchak 2004] and the *parimutuel Kelly probability scoring rule* [Johnstone 2007] are also one-shot budget-balanced betting mechanisms. The competitive scoring rules are incentive compatible but require that all participants wager the same amount of money. While the parimutuel Kelly probability scoring rule can account for different wagers, it loses incentive compatibility.

In spirit, our observation that there are arbitrage opportunities in the WSWMs is analogous to the results of Chun and Shachter [2011]. They showed that when a group of agents is facing a proper scoring rule or participating in a WSWM, they can make an identical prediction and as a coalition obtain a total payoff that is higher than the total payoff they can obtain when each is predicting according to their true beliefs. Their results have the same intuition as ours that participants with immutable beliefs leave free money on the table. In this work, we provide a precise characterization of the arbitrage opportunities in the WSWMs and propose new mechanisms to eliminate the arbitrage and make a profit.

2. PRELIMINARIES

In this section, we introduce preliminaries that are essential for our later analysis. Section 2.1 describes the model that was introduced by Lambert et al. [2008] that this paper adopts. In Section 2.2, we define a set of desirable properties for wagering mechanisms. In Section 2.3, we discuss basics of proper scoring rules that are necessary for understanding prior results and our results on wagering mechanisms, and describe the

WSWMs. Finally, in Section 2.4, we introduce an $(f, \boldsymbol{\mu})$-norm function which is used throughout the paper.

2.1. Wagering Mechanisms

A center is interested in eliciting information on a given random variable X taking value in $\{0, 1\}$. For example, the random variable could represent the outcome of the Super Bowl or whether a user will click on an ad if shown. We focus on binary random variables for this paper because they are common for betting mechanisms and they simplify the analysis, allowing for better insights and intuitions. However, many of our results naturally generalize to any finite discrete random variable.

Consider any finite set of agents, denoted $\mathcal{N} = \{1, 2, \ldots, n\}$. Each agent i forms a subjective belief p_i about the realization of X, where p_i is the probability that agent i assigns to $X = 1$ and hence $1 - p_i$ is his belief on $X = 0$. We use $\mathbf{p}$ to denote the beliefs of all agents and $\mathbf{p}_{-i}$ the beliefs of all agents except agent i. We make no assumption about how agents form their beliefs, and only assume that their beliefs are immutable.

The center uses a *wagering mechanism* to elicit the private beliefs of agents. In a wagering mechanism, each agent makes a prediction $\hat{p}_i \in [0, 1]$ and wagers some non-negative amount of money $w_i \in \mathbf{R}_+$, where $\mathbf{R}_+$ is the set of non-negative reals. We use $\hat{\mathbf{p}}$ and $\mathbf{w}$ to represent the predictions and wagers of all agents and $\hat{\mathbf{p}}_{-i}$ and $\mathbf{w}_{-i}$ to represent the predictions and wagers of all agents except agent i. After the outcome of the random experiment, $x \in \{0, 1\}$, is revealed, the mechanism determines the *net* payoff of each agent, $\Pi_i(\hat{\mathbf{p}}, \mathbf{w}, x) \in \mathbf{R}$, with $\Pi_i(\hat{\mathbf{p}}, \mathbf{w}, x) = 0$ whenever $w_i = 0$.[1] We emphasize that in this paper we will always talk about the net payoff of an agent, that is, the payment received by the agent minus his wager, although all wagering mechanisms can be equivalently defined by describing just the payments that agents receive. It is natural to require that an agent cannot lose more than his wager, that is, $\Pi_i(\hat{\mathbf{p}}, \mathbf{w}, x) \geq -w_i$ for all $\hat{\mathbf{p}} \in [0, 1]^n$, $\mathbf{w} \in \mathbf{R}_+^n$, and $x \in \{0, 1\}$.

The wagers of agents are decided up-front before they participate in the mechanism. For example, an agent may allocate a budget, which depends on his confidence about his information, for wagering on the random experiment. We further assume that agents are risk neutral and maximize the expected net payoff with respect to their belief. Risk neutrality is often observed when the wager of an agent is relatively small with respect to his wealth [Ali 1977].

2.2. Properties of Wagering Mechanisms

Designers of wagering mechanisms often want their mechanisms to satisfy some desirable properties. We introduce and define the set of properties that we consider in this paper below. The first six properties are satisfied by all WSWMs [Lambert et al. 2008], which we review in the next subsection; some, but not all, also obey neutrality. We use $X \sim p$ to represent that the random variable X follows a distribution $\mathrm{P}(X = 1) = p$ and $\mathrm{P}(X = 0) = 1 - p$.

(a) **Anonymity**: The identities of agents do not affect their net payoff in the mechanism. Formally, for any permutation σ of $\mathcal{N}$,

$$\Pi_i(p_1, \ldots, p_n, w_1, \ldots, w_n, x) = \Pi_{\sigma^{-1}(i)}(p_{\sigma(1)}, \ldots, p_{\sigma(n)}, w_{\sigma(1)}, \ldots, w_{\sigma(n)}, x),$$

$\forall\, i \in \mathcal{N}, \mathbf{p} \in [0, 1]^n, \mathbf{w} \in \mathbf{R}_+^n$, and $x \in \{0, 1\}$.

[1] Strictly speaking, the net payoff Π_i also depends on the total number of agents who participate, n. We omit this dependency in our notation as it is clear in the context.

(b) **Budget balance**: The mechanism does not make or lose money. That is, $\forall$ $\mathbf{p} \in [0,1]^n$, $\mathbf{w} \in \mathbf{R}_+^n$, and $x \in \{0,1\}$,

$$\sum_{i \in \mathcal{N}} \Pi_i(\mathbf{p}, \mathbf{w}, x) = 0.$$

(b') **Weak budget balance**: The mechanism does not lose money but can (optionally) make a profit. That is, $\forall$ $\mathbf{p} \in [0,1]^n$, $\mathbf{w} \in \mathbf{R}_+^n$, and $x \in \{0,1\}$,

$$\sum_{i \in \mathcal{N}} \Pi_i(\mathbf{p}, \mathbf{w}, x) \leq 0.$$

(c) **Individual rationality**: Every agent prefers participating in the mechanism to not participating. Formally, $\forall$ $i \in \mathcal{N}$, $p_i \in [0,1]$, and $w_i \in \mathbf{R}_+$, there exists some $\hat{p}_i \in [0,1]$ such that for all $\hat{\mathbf{p}}_{-i} \in [0,1]^{n-1}$, and $\mathbf{w}_{-i} \in \mathbf{R}_+^{n-1}$,

$$E_{X \sim p_i}[\Pi_i(\hat{p}_i, \hat{\mathbf{p}}_{-i}), w_i, \mathbf{w}_{-i}, X)] \geq 0.$$

(d) **Incentive compatibility**: Every agent strictly maximizes his expected net payoff by predicting his true belief. Formally, $\forall$ $i \in \mathcal{N}$, $p_i \in [0,1]$, $\hat{p}_i \in [0,1]$, $\hat{\mathbf{p}}_{-i} \in [0,1]^{n-1}$, and $\mathbf{w} \in \mathbf{R}_+^n$,

$$E_{X \sim p_i}[\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, X)] \geq E_{X \sim p_i}[\Pi_i(\hat{p}_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, X)],$$

and the inequality is strict when $p_i \neq \hat{p}_i$.

(e) **Normality**:[2] If from agent i's perspective the prediction of another agent j improves, then agent i's expected net payoff decreases. Formally, $\forall i \neq j$ and $i, j \in \mathcal{N}, p_i \in [0,1]$, $\hat{\mathbf{p}} \in [0,1]^n$, and $\mathbf{w} \in \mathbf{R}_+^n$, define another prediction vector $\hat{\mathbf{p}}'$ where $\hat{p}'_j = p_i$ and $\hat{p}'_k = \hat{p}_k$ for all $k \in \mathcal{N}$ and $k \neq j$. Then,

$$E_{X \sim p_i}[\Pi_i(\hat{\mathbf{p}}, \mathbf{w}, X)] \geq E_{X \sim p_i}[\Pi_i(\hat{\mathbf{p}}', \mathbf{w}, X)].$$

The inequality is strict when $\hat{p}_j \neq p_i$.

(f) **Sybilproofness**:[3] It is not beneficial for any agent to create fake identities and participate in the mechanism under these identities. Formally, $\forall$ $i \in \mathcal{N}$, integer $k > 1$, $p_i \in [0,1]$, $\hat{\mathbf{p}}_{-i} \in [0,1]^{n-1}$, $w_i \in \mathbf{R}_+$ and $\mathbf{w}_{-i} \in \mathbf{R}_+^{n-1}$, if agent i participates under k identities and p_{i_j} and w_{i_j} are the j-th identity's prediction and wager respectively, where $\sum_{j=1}^k w_{i_j} = w_i$ and $p_{i_j} \in [0,1], \forall j, 1 \leq j \leq k$, there exists a $\hat{p}_i \in [0,1]$ such that

$$E_{X \sim p_i}[\Pi_i(\hat{p}_i, \hat{\mathbf{p}}_{-i}, w_i, \mathbf{w}_{-i}, X)] \geq \sum_{j=1}^{k} E_{X \sim p_i}[\Pi_{i_j}(p_{i_1}, \ldots, p_{i_k}, \hat{\mathbf{p}}_{-i}, w_{i_1}, \ldots, w_{i_k}, \mathbf{w}_{-i}, X)].$$

(g) **Neutrality:** The net payoff of any agent is invariant under relabeling of the outcomes. Formally, $\forall i \in \mathcal{N}$, $\mathbf{p} \in [0,1]^n$, $\mathbf{w} \in \mathbf{R}_+^n$ and $x \in \{0,1\}$,

$$\Pi_i(\mathbf{p}, \mathbf{w}, x) = \Pi_i(\mathbf{1} - \mathbf{p}, \mathbf{w}, 1 - x),$$

where $\mathbf{1} - \mathbf{p}$ is a vector with elements $1 - p_i,\ i \in \mathcal{N}$.

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[2] Lambert et al. [2008] define an alternative notion of normality: if an agent's expected net payoff increases according to a belief p, the expected net payoff of each other player according to this belief decreases. The property we define here is used in Lambert et al. [2014].

[3] This definition is weaker than the definition given in Lambert et al. [2008] which requires that an agent receives *exactly the same* payoff if he splits or merges a bet.

2.3. Proper Scoring Rules and Weighted Score Wagering Mechanisms

Existing wagering mechanisms [Kilgour and Gerchak 2004; Johnstone 2007; Lambert et al. 2008, 2014] use proper scoring rules as building blocks. Thus, before introducing the WSWMs, we first define proper scoring rules and discuss some of their properties.

2.3.1. Proper Scoring Rules. Scoring rules are developed to reward individuals for their predictions on a random variable. They are defined for any discrete or continuous random variable. But for consistency with our settings, we introduce them for binary random variables. A scoring rule $s(p, x)$ maps a prediction and an outcome to a value in $\mathbf{R} \cup \{-\infty\}$. Here p represents an individual's prediction on the probability of outcome 1. For notational convenience, we use $s_x(p)$ to denote $s(p, x)$. In order to avoid an individual's expected score going to $-\infty$, a regularity condition is often required. A scoring rule is regular if it takes value $-\infty$ only if the prediction for the corresponding outcome is 0. For binary outcomes, this means that it is only possible for $s_1(0)$ and $s_0(1)$ to have value $-\infty$. A strictly proper scoring rule, as defined below, strictly incentivizes a risk-neutral individual to predict according to his true belief.

Definition 2.1. A regular scoring rule s is *strictly proper* if and only if for all $p, q \in [0, 1]$ and $p \neq q$, $E_{X\sim p}[s_X(p)] > E_{X\sim p}[s_X(q)]$.

A widely used strictly proper scoring rule is the Brier scoring rule [Brier 1950]:

$$s_x^B(p) = 1 - (p - x)^2, \tag{1}$$

for $x \in \{0, 1\}$.

Strictly proper scoring rules are closely related to strictly convex functions. In fact, Savage [1971] and Gneiting and Raftery [2007] characterized all scoring rules in terms of subdifferentials of convex functions. The version for binary random variables is stated below.

THEOREM 2.2. *(Savage [1971] and Gneiting and Raftery [2007]) Every strictly proper scoring rule s can be represented as*

$$s_x(p) = G(p) + (x - p)G'(p), \quad \forall x \in \{0, 1\}$$

for some strictly convex function G, where $G'(p)$ is a subgradient of G at p. Moreover, $G(p) = E_{X\sim p}[s_X(p)]$.

It is easy to show that $s_1(p)$ monotonically increases and $s_0(p)$ monotonically decreases with p if s is strictly proper and differentiable.

COROLLARY 2.3. *If a strictly proper scoring rule $s_x(p)$ is differentiable for all $x \in \{0, 1\}$, then $s_1(p)$ is a strictly increasing function of p and $s_0(p)$ is a strictly decreasing function of p.*

2.3.2. Weighted Score Wagering Mechanisms. Equipped with an understanding of proper scoring rules, we are ready to define weighted score wagering mechanisms. A weighted score wagering mechanism uses a strictly proper scoring rule s. A participant's net payoff in the mechanism depends on how his score compares with the wager-weighted average score of all agents. Let $W_{\mathcal{A}}$ denote the total wager of participants in set $\mathcal{A}$. The net payoff function of weighted score wagering mechanisms is defined as follows.

Definition 2.4. A *weighted score wagering mechanism (WSWM)* with strictly proper scoring rule s, where $s_x(p) \in [0, 1]$ $\forall p \in [0, 1]$ and $x \in \{0, 1\}$, determines the net payoffs of participants according to function

$$\Pi_i^{\mathrm{WS}}(\mathbf{p},\mathbf{w},x) = w_i\left(s_x(p_i) - \sum_{j\in\mathcal{N}}\frac{w_j}{W_{\mathcal{N}}}s_x(p_j)\right)$$
$$= \frac{w_i W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}}}\left(s_x(p_i) - \sum_{j\in\mathcal{N}\setminus\{i\}}\frac{w_j}{W_{\mathcal{N}\setminus\{i\}}}s_x(p_j)\right). \tag{2}$$

The proper scoring rules used in WSWMs are restricted to have range $[0,1]$. This guarantees that no agent can lose more than what he wagers in the mechanism, that is, $\Pi_i^{\mathrm{WS}}(\mathbf{p},\mathbf{w},x) \geq -w_i$. We also often assume that s is differentiable; in such cases we refer to the mechanism as differentiable WSWMs.

WSWMs have been shown to satisfy anonymity, budget balance, individual rationality, incentive compatibility, normality, and a stronger version of sybilproofness [Lambert et al. 2008].[4] Lambert et al. [2008] also proved that the WSWMs are the unique set of mechanisms that satisfy the above six properties as well as another homogeneity property, which requires that multiplying all wagers by a positive constant α results in the net payoffs of all agents being multiplied by α. Depending on the choice of the scoring rule, some but not all WSWMs satisfy neutrality.

2.4. f-norm

Finally, we introduce an $(f,\boldsymbol{\mu})$-norm function and derive its properties. These are important technical tools that we will use throughout the paper to understand incentive properties of wagering mechanisms and design new mechanisms.

Let $f:[0,1]\to\mathbf{R}$ be a continuous, strictly monotone function, and let f^{-1} denote the inverse function of f, which is well defined in the range of f. Define the "average" function for a vector $\mathbf{p}\in[0,1]^n$ with respect to a vector of weights $\boldsymbol{\mu}$, with $\sum_i \mu_i = 1$ and $\mu_i \geq 0\ \forall i$, as

$$\boldsymbol{\mu}\text{–avg}(\mathbf{p}) := \sum_{i=1}^n \mu_i p_i.$$

We abuse notation to let $f(\mathbf{p})$ denote the vector whose i-th component is $f(p_i)$. Define the $(f,\boldsymbol{\mu})$-norm of a vector $\mathbf{p}\in[0,1]^n$ with respect to $\boldsymbol{\mu}$ as follows:

$$\|\mathbf{p}\|_{f,\boldsymbol{\mu}} := f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(\mathbf{p}))\right).$$

We will call it f-norm for convenience. We now note some properties of the f-norm.

LEMMA 2.5. *For any continuous, strictly monotone function $f:[0,1]\to\mathbf{R}$, any vector of weights $\boldsymbol{\mu}$ and any constants a and b,*

(1) *If $g(x) = af(x)+b$, then $\|\mathbf{p}\|_{g,\boldsymbol{\mu}} = \|\mathbf{p}\|_{f,\boldsymbol{\mu}}$.*
(2) *If $g(x) = f(ax+b)$, then $\|\mathbf{p}\|_{g,\boldsymbol{\mu}} = \frac{\|a\mathbf{p}+b\|_{f,\boldsymbol{\mu}}-b}{a}$.*

LEMMA 2.6. *Let f and g be continuous, strictly monotone functions.*

$$\|\mathbf{p}\|_{f,\boldsymbol{\mu}} \leq \|\mathbf{p}\|_{g,\boldsymbol{\mu}}\ \forall\,\boldsymbol{\mu} \Leftrightarrow \begin{cases} g(f^{-1}(\cdot)) \text{ is convex} & \text{if } g \text{ is increasing} \\ g(f^{-1}(\cdot)) \text{ is concave} & \text{if } g \text{ is decreasing.}\end{cases}$$

[4]While the sybilproofness property in Section 2.2 ensures that a participant won't find splitting his wager and betting under multiple identities profitable *ex ante*, WSWMs guarantee it even *ex post* because the net payoff of each participant remains the same for all x. For the same reason, WSWMs also ensure that participants with the same prediction do not find it profitable to pool their wagers and participate under a single identity.

3. ARBITRAGE IN WEIGHTED SCORE WAGERING MECHANISMS

We show in this section that the appealing WSWMs leave free money on the table. In other words, there exist arbitrage opportunities in these mechanisms. We formalize this notion of an arbitrage, characterize the *arbitrage interval* for WSWMs and give sufficient conditions for any wagering mechanism to have an arbitrage opportunity.

We first demonstrate the existence of an arbitrage opportunity in the Brier wagering mechanism, the WSWM with the Brier scoring rule. In the Brier wagering mechanism, given prediction vector $\mathbf{p}$ and wager vector $\mathbf{w}$, participant i receives net payoff, as defined in (2),

$$\Pi_i^{\mathrm{WS}}(\mathbf{p}, \mathbf{w}, x) = \frac{w_i W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}}} \left(-(x-p_i)^2 + \sum_{j\in\mathcal{N}\setminus\{i\}} \frac{w_j}{W_{\mathcal{N}\setminus\{i\}}}(x-p_j)^2 \right)$$

under outcome x. Now suppose participant i instead makes a prediction

$$\hat{p}_i = \sum_{j\in\mathcal{N}\setminus\{i\}} \frac{w_j}{W_{\mathcal{N}\setminus\{i\}}} p_j.$$

Then, his net payoff is nonnegative due to the convexity of the function $f(p) = (x-p)^2$, and is strictly positive if not all $p_j,\ j \in \mathcal{N}\setminus\{i\}$, take the same value. Further, it is easy to see that the value of $\Pi_i^{\mathrm{WS}}(\hat{p}_i, \mathbf{p}_{-i}, \mathbf{w}, x)$ is the same for any outcome x. This means that by predicting $\hat{p}_i$, participant i makes a strictly positive and outcome-independent net payoff whenever there exists disagreement among other participants.

The opportunity for a participant to *risklessly make a profit* is what we call an arbitrage opportunity. We now formally define what risklessly making a profit means in a wagering mechanism.

Definition 3.1. A participant i *risklessly makes a profit* at predictions $\mathbf{p}$ and wagers $\mathbf{w}$ in a wagering mechanism if and only if both $\Pi_i(\mathbf{p}, \mathbf{w}, 1)$ and $\Pi_i(\mathbf{p}, \mathbf{w}, 0)$ are nonnegative and at least one of $\Pi_i(\mathbf{p}, \mathbf{w}, 1)$ and $\Pi_i(\mathbf{p}, \mathbf{w}, 0)$ is strictly positive.

The existence of arbitrage opportunities does not contradict the incentive compatibility of the mechanism. This is because participant i still maximizes his expected net payoff by predicting his true belief p_i but he may receive a negative net payoff for one outcome with this prediction. In addition, given the one-shot nature of the mechanism, participants seeking arbitrage have no way of knowing $\hat{p}_i$.

3.1. Characterization of Arbitrage Interval for WSWMs

The demonstrated arbitrage opportunity is not unique to the Brier wagering mechanism and can be generalized to any WSWM. For this, it's useful to restate the above observation in terms of f-norms. Notice that we set $\hat{p}_i = \|\mathbf{p}_{-i}\|_{f,\boldsymbol{\mu}}$ where $f(x) = x$ is the identity function and $\mu_j = w_j/W_{\mathcal{N}\setminus\{i\}}$ for all $j \neq i$. For an arbitrary scoring rule s, let G be the corresponding convex function in Theorem 2.2. Then the following lemma gives the corresponding equal arbitrage prediction point for a participant. The proof of the lemma is very similar to that of Theorem 1 in Chun and Shachter [2011], where they show that a group of agents can "collude" by all making a prediction at this point and, as a coalition, make more profit under every outcome than everyone predicting his true belief. We omit the proof here.

LEMMA 3.2. *For any differentiable WSWM with scoring rule s, a participant i risklessly makes a profit at prediction $\hat{p}_i = \|\mathbf{p}_{-i}\|_{G',\boldsymbol{\mu}}$, where G' is the gradient of $G(p) = E_{X\sim p}[s_X(p)]$ and $\mu_j = w_j/W_{\mathcal{N}\setminus\{i\}}$ for all $j \in \mathcal{N}\setminus\{i\}$, as long as not all elements* *in $\mathbf{p}_{-i}$ are the same (in which case his payoff is 0). Further, his payoff is the same for all outcomes.*

We can further characterize the entire interval that allows a participant to risklessly make a profit.

THEOREM 3.3. *For any differentiable WSWM with scoring rule s, a participant i risklessly makes a profit at predictions $\mathbf{p}$ and wagers $\mathbf{w}$ if and only if*

(*1*) $p_i \in [\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}, \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}]$ *where* $\mu_j = w_j/W_{\mathcal{N}\setminus\{i\}}$ *for all* $j \in \mathcal{N}\setminus\{i\}$*, and*
(*2*) *not all elements in* $\mathbf{p}_{-i}$ *are the same.*

PROOF. We first prove the "if part." Suppose that not all elements in $\mathbf{p}_{-i}$ are the same. Since the scoring rule s is strictly proper and differentiable, according to Corollary 2.3, $s_1(p)$ is a strictly increasing function and $s_0(p)$ is a strictly decreasing function. From (2), one can see that participant i's net payoff $\Pi_i^{\text{WS}}(\mathbf{p},\mathbf{w},1)$ strictly increases with p_i and $\Pi_i^{\text{WS}}(\mathbf{p},\mathbf{w},0)$ strictly decreases with p_i. From Lemma 3.2, there exists a point $\hat{p}_i$ for which both payments are positive. Suppose participant i predicts

$$\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}} = s_1^{-1}\left(\sum_{j\in\mathcal{N}\setminus\{i\}} \frac{w_j}{W_{\mathcal{N}\setminus\{i\}}} s_1(p_j)\right).$$

Then

$$\Pi_i^{\text{WS}}(\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}, \mathbf{p}_{-i}, \mathbf{w}, 1) = s_1(\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}) - \sum_{j\in\mathcal{N}\setminus\{i\}} \frac{w_j}{W_{\mathcal{N}\setminus\{i\}}} s_1(p_j) = 0.$$

Since $\Pi_i^{\text{WS}}(\mathbf{p},\mathbf{w},1)$ strictly increases with p_i, we must have $\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}} < \hat{p}_i$. Similarly, we can show that $\Pi_i^{\text{WS}}(\|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}, \mathbf{p}_{-i}, \mathbf{w}, 0) = 0$ and $\|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}} > \hat{p}_i$. Again using the monotonicity of payoffs, it is easy to see that the payoff for any prediction in the nonempty range $[\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}, \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}]$ results in a nonnegative payoff for trader i for both outcomes and a strictly positive payoff for at least one.

Now for the "only if" part, when not all elements in $\mathbf{p}_{-i}$ are the same, the above analysis already shows that if $p_i \notin [\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}, \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}]$, then either $\Pi_i^{\text{WS}}(p_i,\mathbf{p}_{-i},\mathbf{w},0)$ or $\Pi_i^{\text{WS}}(p_i,\mathbf{p}_{-i},\mathbf{w},1)$ is strictly negative, and hence participant i cannot risklessly make a profit. Suppose that all elements in $\mathbf{p}_{-i}$ are the same, and are equal to p. Then

$$\Pi_i^{\text{WS}}(\mathbf{p},\mathbf{w},x) = \frac{w_i W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}}}\left(s_x(p_i) - s_x(p)\right).$$

Participant i can then never risklessly make a profit, due to the strict monotonicity of s_1 and s_0. If $p_i > p$, then $s_0(p_i) < s_0(p)$ and if $p_i < p$, then $s_1(p_i) < s_1(p)$, so in either case participant i could make a loss for one of the outcomes. If $p_i = p$ then she makes 0 for both outcomes. □

3.2. Sufficient Conditions for Existence of Arbitrage

Finally, we show that the existence of arbitrage opportunities seems to be quite general for wagering mechanisms by providing some mild sufficient conditions. We make no assumptions about the wagering mechanisms aside from the three conditions listed in the theorem statement; in particular, we do not assume weak budget balance, individual rationality, or even incentive compatibility.

THEOREM 3.4. *If a wagering mechanism satisfies the following three conditions:*

(*1*) $\forall\, i \in \mathcal{N}$*,* $x \in \{0,1\}$*,* $\mathbf{p}_{-i} \in [0,1]^{n-1}$*, and* $\mathbf{w} \in \mathbf{R}^n$*,* $\Pi_i(p_i,\mathbf{p}_{-i},\mathbf{w},x)$ *is continuous in* p_i*.*

(2) *If* $p_i = p_j$ *for all* $i, j \in \mathcal{N}$*, then* $\Pi_i(\mathbf{p}, \mathbf{w}, x) \geq 0$ *for all* $i \in \mathcal{N}$*,* $\mathbf{w} \in \mathbf{R}^n$*, and* $x \in \{0, 1\}$.
(3) *The normality condition (as defined in Section 2.2).*

then for all $i \in \mathcal{N}$*,* $\mathbf{w} \in \mathbf{R}^n_+$*, and* $\hat{\mathbf{p}}_{-i} \in [0,1]^{n-1}$ *where not all elements of* $\hat{\mathbf{p}}_{-i}$ *are the same, there exists an interval* $[a_i, b_i]$*, where* $0 \leq a_i < b_i \leq 1$*, such that agent* i *risklessly makes a profit by predicting some* $\hat{p}_i \in [a_i, b_i]$.

PROOF. Normality says that for all $p_i \in [0,1]$ and $\mathbf{w} \in \mathbf{R}^n_+$, holding any $\hat{\mathbf{p}}_{-j} \in [0,1]^{n-1}$ fixed, $j \neq i$, the expected payoff of agent i with belief p_i is strictly minimized when agent j predicts $\hat{p}_j = p_i$, that is, $p_i = \arg\min_{\hat{p}_j} E_{X \sim p_i}[\Pi_i(\hat{p}_j, \hat{\mathbf{p}}_{-j}, \mathbf{w}, X)]$.

We first prove that conditions (2) and (3) together imply the following:

(*) For all $i \in \mathcal{N}$, belief $p_i \in [0,1]$, predictions $\hat{\mathbf{p}}_{-i} \in [0,1]^{n-1}$ such that not all elements of $\hat{\mathbf{p}}_{-i}$ are the same, and wagers $\mathbf{w} \in \mathbf{R}^n_+$, the expected net payoff of agent i is strictly positive if he predicts p_i, that is, $E_{X \sim p_i}[\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, X)] > 0$.

To show this, consider $E_{X \sim p_i}[\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, X)]$ and $E_{X \sim p_i}[\Pi_i(p_i, \bar{\mathbf{p}}_{-i}, \mathbf{w}, X)]$ where every element of $\bar{\mathbf{p}}_{-i}$ equals p_i. According to condition (2), $E_{X \sim p_i}[\Pi_i(p_i, \bar{\mathbf{p}}_{-i}, \mathbf{w}, X)] \geq 0$. Because not all elements of $\hat{\mathbf{p}}_{-i}$ are the same, there are predictions in $\hat{\mathbf{p}}_{-i}$ that are different from p_i. Now, change those predictions to p_i one at a time until we have $\bar{\mathbf{p}}_{-i}$. At each step, according to condition (3), the expected net payoff of agent i strictly decreases. Combining with $E_{X \sim p_i}[\Pi_i(p_i, \bar{\mathbf{p}}_{-i}, \mathbf{w}, X)] \geq 0$, this means that $E_{X \sim p_i}[\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, X)] > 0$.

Next, we prove that (1) and (*) imply the existence of arbitrage opportunities.

For all $\mathbf{w} \in \mathbf{R}^n_+$ and $\hat{\mathbf{p}}_{-i} \in [0,1]^{n-1}$ such that not all elements of $\hat{\mathbf{p}}_{-i}$ are the same, (*) requires that

$$p_i \Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) + (1 - p_i)\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0) > 0$$

for all $p_i \in [0,1]$. Considering the cases in which $p_i = 1$ and $p_i = 0$, this gives us $\Pi_i(1, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) > 0$ and $\Pi_i(0, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0) > 0$.

According to condition (1), both $\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1)$ and $\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0)$ are continuous functions of p_i. If $\Pi_i(1, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0) > 0$, then the net payoff of agent i is strictly positive under both outcomes when he predicts 1. By continuity of $\Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1)$, there exists a positive ϵ such that when agent i's prediction is in $[1-\epsilon, 1]$, he risklessly makes a profit. By a similar argument, if $\Pi_i(0, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) > 0$, there exists an interval of predictions where agent i risklessly makes a profit by making a prediction in the interval.

The only case left is both $\Pi_i(1, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0) \leq 0$ and $\Pi_i(0, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) \leq 0$. For this case, define $h(p_i) = \Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) - \Pi_i(p_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0)$. $h(p_i)$ is a continuous function defined on $[0,1]$, with $h(1) > 0$ and $h(0) < 0$. By the intermediate value theorem of continuous functions, there exists a $p'_i \in (0,1)$ such that $h(p'_i) = 0$. Because we must have either $\Pi_i(p'_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) > 0$ or $\Pi_i(p'_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0) > 0$, $h(p'_i) = 0$ means that $\Pi_i(p'_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 1) = \Pi_i(p'_i, \hat{\mathbf{p}}_{-i}, \mathbf{w}, 0) > 0$. Agent i makes a strictly positive profit under both outcomes when predicting p'_i. By continuity of the net payoff functions, this implies that there exists $\epsilon > 0$ such that when agent i predicts $\hat{p}_i \in [p'_i - \epsilon, p'_i + \epsilon]$, he risklessly makes a profit. □

The WSWMs, which are normal, satisfy condition (2) in Theorem 3.4 because everyone receives zero net payoff when all agents make the same prediction. It is interesting to note that any wagering mechanism that is anonymous, individually rational, and weakly budget balanced necessarily assigns zero net payoff to all agents when they have the same prediction and wager, that is, condition (2) is satisfied whenever $\mathbf{w}$ has identical elements. Therefore, it is not practical to relax this condition. In the next two sections, we show that it is possible to relax the normality condition and design mechanisms that do not admit arbitrage opportunities yet still satisfy anonymity, weak budget balance, individual rationality, incentive compatibility, and sybilproofness.

4. NO-ARBITRAGE WAGERING MECHANISMS

In this section, we consider mechanisms that plug the arbitrage hole in WSWMs and allow the center to make a profit. We formally define the *no arbitrage* property, and give necessary and/or sufficient conditions for a mechanism to satisfy no arbitrage and/or weak budget balance. In particular, we define a class of mechanisms called no-arbitrage wagering mechanisms that are closely related to WSWMs, but with the no arbitrage property.

No Arbitrage: For all $i \in \mathcal{N}$, $\mathbf{p} \in [0,1]^n$, and $\mathbf{w} \in \mathbf{R}_+^n$, participant i cannot risklessly make a profit at $\mathbf{p}$ and $\mathbf{w}$.

We first provide a characterization theorem on the common form of wagering mechanisms that satisfy individual rationality, incentive compatibility, and no arbitrage. We continue the convention and call a wagering mechanism differentiable if its net payoff function is differentiable with respect to the predictions of the agents.

THEOREM 4.1. *A differentiable wagering mechanism satisfies individual rationality, incentive compatibility, and no arbitrage if and only if its net payoff function is of the form*

$$\Pi_i(\mathbf{p}, \mathbf{w}, x) = c_i(\mathbf{p}_{-i}, \mathbf{w})[s_x(p_i) - s_x(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}))], \tag{3}$$

where s is a strictly proper scoring rule, c_i and $\bar{p}_i$ are functions of only $\mathbf{p}_{-i}$ *and* $\mathbf{w}$*, and* $c_i(\mathbf{p}_{-i}, \mathbf{w}) > 0$ *and* $\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}) \in [0,1]$ *for all* $\mathbf{p}_{-i} \in [0,1]^{n-1}$ *and* $\mathbf{w} \in \mathbf{R}_+^n$.

PROOF. We first prove the sufficiency of (3). Because s is a strictly proper scoring rule and c is a strictly positive function that doesn't not depend on p_i, by definition of strictly proper scoring rules, a wagering mechanism with a net payoff function of (3) is incentive compatible. It's easy to see that it is also individually rational because strict properness of s implies $E_{X\ p_i}[s_X(p_i)] \geq E_{X\ p_i}[s_X(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}))]$. The proof of no arbitrage follows essentially the same argument in the "only if" part of the proof of Theorem 3.3. Basically, if $p_i \neq \bar{p}_i(\mathbf{p}_{-i}, \mathbf{w})$, then either $s_1(p_i) < s_1(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}))$ or $s_0(p_i) < s_0(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}))$ by monotonicity of the scoring functions.

Next, we prove the necessity of (3).

Incentive compatibility requires that, when fixing $\mathbf{p}_{-i}$ and $\mathbf{w}$, the net payoff of agent i is a strictly proper scoring rule of the prediction of i. Hence, it can be written as

$$\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, x) = c_i(\mathbf{p}_{-i}, \mathbf{w}) s_x(p_i) + h_i(\mathbf{p}_{-i}, \mathbf{w}, x) \tag{4}$$

where s is some strictly proper scoring rule and c_i is a strictly positive function. That is, $\mathbf{p}_{-i}$ and $\mathbf{w}$ only affect the affine transformation of some strictly proper scoring rule. Since s is a strictly proper scoring rule and Π_i is differentiable, according to Corollary 2.3, $s_1(p_i)$ is strictly increasing with p_i and $s_0(p_i)$ is strictly decreasing with p_i. This implies that $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 1)$ is strictly increasing with p_i and $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 0)$ is strictly decreasing with p_i.

No arbitrage implies that for all p_i, $\mathbf{p}_{-i}$ and $\mathbf{w}$, either $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 1) = \Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 0) = 0$ or at least one of $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 1)$ and $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 0)$ is strictly negative. By individual rationality and incentive compatibility, we know that for any p_i, $E_{X\sim p_i}[\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, X)] \geq 0$. This means that for all p_i either $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 1) = \Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 0) = 0$ or one of $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 1)$ and $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 0)$ is at least 0 and the other is strictly negative. Moreover, because $E_{X\sim 1}[\Pi_i(1, \mathbf{p}_{-i}, \mathbf{w}, X)] = \Pi_i(1, \mathbf{p}_{-i}, \mathbf{w}, 1)$ and $E_{X\sim 0}[\Pi_i(0, \mathbf{p}_{-i}, \mathbf{w}, X)] = \Pi_i(0, \mathbf{p}_{-i}, \mathbf{w}, 0)$, we know that $\Pi_i(1, \mathbf{p}_{-i}, \mathbf{w}, 1) \geq 0$ and $\Pi_i(0, \mathbf{p}_{-i}, \mathbf{w}, 0) \geq 0$. Together with the monotonicity of $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 1)$ and $\Pi_i(p_i, \mathbf{p}_{-i}, \mathbf{w}, 0)$, all of these imply that there exists some $\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}) \in [0,1]$ such that $\Pi_i(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}), \mathbf{p}_{-i}, \mathbf{w}, 1) = \Pi_i(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}), \mathbf{p}_{-i}, \mathbf{w}, 0) = 0$. Applying (4), this implies

$$c_i(\mathbf{p}_{-i}, \mathbf{w}) s_x(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w})) + h_i(\mathbf{p}_{-i}, \mathbf{w}, x) = 0.$$

This gives that $h_i(\mathbf{p}_{-i}, x) = -c_i(\mathbf{p}_{-i}, \mathbf{w}) s_x(\bar{p}_i(\mathbf{p}_{-i}, \mathbf{w}))$. Plugging this to (4) gives (3). $\square$

4.1. No-Arbitrage Wagering Mechanisms and Weak Budget Balance

Theorem 4.1 gives quite some flexibility in selecting c_i and $\bar{p}_i$. In this section, we hope to design wagering mechanisms that not only satisfy the incentive properties but also allow the center to make a profit. This means that we would like to have wagering mechanisms satisfying weak budget balance. Intuitively, we'd like to remove arbitrage opportunities from WSWMs, but in a way that benefits the center. It is interesting to note that wagering mechanisms with net payoff functions in (3) may not necessarily allow the center to make a profit even if they don't admit arbitrage opportunities.

We choose to focus on a subset of the mechanisms characterized by Theorem 4.1 because, as it will become evident soon, this subset of wagering mechanisms connects to WSWMs in a natural way. We call this family of wagering mechanisms the *no-arbitrage wagering mechanisms (NAWM)* and define it formally below. Given a permutation σ of agents in $\mathcal{N}\backslash\{i\}$, we use $\mathbf{p}^{\sigma}_{-i}$ and $\mathbf{w}^{\sigma}_{-i}$ to denote the vectors achieved by permuting elements of $\mathbf{p}_{-i}$ and $\mathbf{w}_{-i}$ according to σ respectively.

Definition 4.2. A *no-arbitrage wagering mechanism (NAWM)* determines the net payoffs of agents according to function

$$\Pi_i^{\mathrm{NA}}(\mathbf{p}, \mathbf{w}, x) = \frac{w_i W_{\mathcal{N}\backslash\{i\}}}{W_{\mathcal{N}}} \left[s_x(p_i) - s_x\left(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})\right)\right], \tag{5}$$

where s is any strictly proper scoring rule such that $s_x(p) \in [0,1]$ for all $p \in [0,1]$ and $x \in \{0,1\}$, and $\bar{p}$ is any function such that $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) \in [0,1]$ and $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) = \bar{p}(\mathbf{p}^{\sigma}_{-i}, \mathbf{w}^{\sigma}_{-i})$ for all $\mathbf{p}_{-i} \in [0,1]^{n-1}$, $\mathbf{w}_{-i} \in \mathbf{R}_+^{n-1}$, and all permutations σ of $\mathcal{N}\backslash\{i\}$.

NAWMs restrict that c_i in (3) is the same function for all agents and depends on neither the predictions of other agents nor the identities of agents. It is the same multiplier that appears in the net payoff functions of WSWMs. The definition also requires that $\bar{p}_i$ in (3) is the same function for all agents and doesn't depend on the identities of the agents or the wager of agent i. Thus, by definition, NAWMs satisfy anonymity, and if differentiable, by Theorem 4.1, they satisfy individual rationality, incentive compatibility, and no arbitrage.

COROLLARY 4.3. *Any differentiable NAWM satisfies individual rationality, incentive compatibility, anonymity, and no arbitrage.*

We sometimes use $\bar{p}_i$ to denote $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ when the vectors of predictions and wagers are clear in the context. It is easy to check that the net payoff functions of NAWMs can be written as

$$\Pi_i^{\mathrm{NA}}(\mathbf{p}, \mathbf{w}, x) = \Pi_i^{\mathrm{WS}}(\mathbf{p}, \mathbf{w}, x) - \Pi_i^{\mathrm{WS}}(\bar{p}_i, \mathbf{p}_{-i}, \mathbf{w}, x). \tag{6}$$

This means that an NAWM works by subtracting some value that is independent of the prediction of agent i from agent i's net payoff in the corresponding WSWM. Since the subtracted value doesn't depend on agent i's prediction, NAWMs preserve the incentive compatibility of WSWMs. If we interpret $\bar{p}_i$ as a prediction that an arbitrager makes when wagering w_i against all agents except i, we observe below that the conditions in Theorem 3.3 are sufficient for an NAWM to satisfy weak budget balance. The proof follows from the observation in (6) and the budget balance of WSWMs.

THEOREM 4.4. *A differentiable NAWM satisfies weak budget balance (in addition to the properties in Corollary 4.3) if for all $i \in \mathcal{N}$, $\mathbf{p} \in [0,1]^n$, and $\mathbf{w} \in \mathbf{R}_+^n$, it satisfies $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) \in [\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}, \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}]$ where $\mu_j = w_j / W_{\mathcal{N}\backslash\{i\}}$ for all $j \in \mathcal{N}\backslash\{i\}$.*

We note that NAWMs violate the normality property. Normality requires that, fixing everything else, if agent j changes his prediction to agent i's true belief, then agent i's expected net payoff is minimized. In NAWMs, depending on the prediction of other agents, agent i's expected net payoff can increase with such a move. To see this, consider an NAWM using the Brier scoring rule (1) and $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) = \sum_j \frac{w_j}{W_{\mathcal{N}\setminus\{i\}}} p_j$. As we showed at the beginning of Section 3, this $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ satisfies the condition in Theorem 4.4. Suppose there are only three agents who predict 0.1, 0.4 and 0.7 respectively and have the same wager. The true belief of agent 2 is also 0.4. With these predictions, $\bar{p}_2 = 0.4$, the same as agent 2's prediction, which leads to a zero net payoff for agent 2 in the NAWM. However, if agent 1 changes her report to 0.4, $\bar{p}_2$ becomes 0.55. Agent 2 now in expectation makes a positive net payoff.

4.2. Adding Neutrality

At this point, we are equipped with a class of wagering mechanisms, characterized by Theorem 4.4, that satisfy anonymity, individual rationality, incentive compatibility, no arbitrage, and weak budget balance. Just as not all WSWMs are neutral, not all NAWMs satisfy neutrality. Next we provide conditions that are necessary and sufficient for an NAWM to satisfy neutrality. In addition to being a desirable property for scenarios in which it would be unnatural for the wagering mechanism to treat outcomes asymmetrically (e.g., when wagering over which of two candidates will win an election), in the next section we will see that neutrality helps us to focus on a smaller set of NAWMs for which we can obtain the explicit functional forms of the payoff functions and analyze the property of sybilproofness.

We extend the definition of neutrality for a wagering mechanism to scoring rules as well as the function $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$. We say that a scoring rule s is neutral if for all $x \in \{0,1\}$ and $p_i \in [0,1]$, $s_x(p_i) = s_{1-x}(1-p_i)$. We say that the function $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ is neutral if $\bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w}_{-i}) = 1 - \bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ for all $\mathbf{p}_{-i} \in [0,1]^{n-1}$ and $\mathbf{w}_{-i} \in \mathbf{R}_+^{n-1}$.

LEMMA 4.5. *An NAWM satisfies neutrality if and only if*

(1) its net payoff function (5) can be represented with a neutral scoring rule, and
(2) $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ is neutral.

PROOF. Conditions (1) and (2) imply

$$\begin{aligned}\Pi_i^{\text{NA}}(\mathbf{1} - \mathbf{p}, \mathbf{w}, 1-x) &= \frac{w_i W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}}} \left[s_{1-x}(1-p_i) - s_{1-x}(\bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w}_{-i}))\right] \\ &= \frac{w_i W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}}} \left[s_x(p_i) - s_x(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}))\right] = \Pi_i^{\text{NA}}(\mathbf{p}, \mathbf{w}, x).\end{aligned}$$

Hence the mechanism is neutral.

Next we prove that conditions (1) and (2) are necessary for an NAWM to satisfy neutrality. Neutrality of an NAWM requires

$$s_x(p_i) - s_x(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})) = s_{1-x}(1-p_i) - s_{1-x}(\bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w}_{-i})) \tag{7}$$

for all $\mathbf{p}$, $\mathbf{w}$, and x.

We use $\mathbf{0.5}$ to represent a vector whose elements are 0.5. Now consider the case where $\mathbf{p}_{-i} = \mathbf{0.5}$. Clearly, $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) = \bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w}_{-i})$ for $\mathbf{p}_{-i} = \mathbf{0.5}$. (7) implies that

$$s_x(p_i) - s_x(\bar{p}(\mathbf{0.5}, \mathbf{w}_{-i})) = s_{1-x}(1-p_i) - s_{1-x}(\bar{p}(\mathbf{0.5}, \mathbf{w}_{-i}))$$

for all p_i, $\mathbf{w}_{-i}$, and x. Pick an arbitrary $\mathbf{w}^*_{-i}$ and define $s'_x(p_i) = s_x(p_i) - s_x(\bar{p}(\mathbf{0.5}, \mathbf{w}^*_{-i}))$. The above expression implies that s' is neutral. It's also easy to check that $s'_x(p_i) - s'_x(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})) = s_x(p_i) - s_x(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}))$ for all $\mathbf{p}$, $\mathbf{w}_{-i}$ and x, which means that the original net payoff function can be written as

$$\Pi_i^{\mathrm{NA}}(\mathbf{p}, \mathbf{w}, x) = \frac{w_i W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}}} \left[s_x'(p_i) - s_x'\left(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})\right) \right].$$

This proves that (1) is necessary.

Letting $p_i = 0.5$ in (7), the neutrality of s' implies that $s_x'(\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})) = s_{1-x}'(\bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w}_{-i}))$ for all $\mathbf{p}_{-i}$ and $\mathbf{w}_{-i}$. Applying the neutrality of s' again, we have $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}) = 1 - \bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w})$, which shows the necessity of (2). □

Since the payoff function of any neutral NAWM can be represented using a neutral scoring rule, we assume without loss of generality that the scoring rule used in any neutral NAWM is also neutral. If an NAWM satisfies neutrality, the conditions in Theorem 4.4 can be slightly weakened, to only require $\bar{p}_i$ to be bounded above by $\|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}$ for weak budget balance to be satisfied, as shown in the following theorem.

THEOREM 4.6. *A differentiable and neutral NAWM satisfies weak budget balance (in addition to the properties in Corollary 4.3) if for all $i \in N$, $\mathbf{w} \in \mathbf{R}_+^n$, and $\mathbf{p} \in [0,1]^n$, $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) \le \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}$ where $\mu_j = w_j / W_{\mathcal{N}\setminus\{i\}}$ for all $j \in \mathcal{N}\setminus\{i\}$.*

In the next section we will see that considering neutrality allows us to restrict our attention to a smaller set of NAWMS that we can analyze more deeply.

5. NO-ARBITRAGE WAGERING MECHANISMS USING f-NORMS

In the previous section we showed that if an NAWM mechanism uses a neutral scoring rule s and a neutral $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ function, then if $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) \le \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}$ is satisfied, the mechanism satisfies neutrality and weak budget balance, in addition to anonymity, individual rationality, incentive compatibility, and no arbitrage. However, we haven't shown what functional forms of $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ satisfy these conditions. Moreover, we don't know whether any of these mechanisms satisfy sybilproofness.

In this section, we propose a generic way of defining $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ that satisfies these conditions, using f-norms. We consider the class of NAWMs that use an f-norm to define $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$. By Corollary 4.3, any differentiable NAWM in this class satisfies anonymity, individual rationality, incentive compatibility, and no arbitrage. We characterize the functions f for which these mechanisms also satisfy weak budget balance and neutrality. We then give specific examples of f-norms that satisfy these properties, and show that these mechanisms also satisfy sybilproofness.

Definition 5.1. For any continuous, strictly monotone function $f : [0,1] \to [0,1]$, an f-NAWM is a NAWM with

$$\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) = \|\mathbf{p}_{-i}\|_{f,\boldsymbol{\mu}},$$

where $\mu_j = w_j / W_{\mathcal{N}\setminus\{i\}}$ for all $j \in \mathcal{N}\setminus\{i\}$.

We first give necessary and sufficient conditions on f for $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ to be neutral, which will allow us to apply the results from Section 4.2.

LEMMA 5.2. *Let $\mu_j = w_j / W_{\mathcal{N}\setminus\{i\}}$ for all $j \in \mathcal{N}\setminus\{i\}$. Then $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) = \|\mathbf{p}_{-i}\|_{f,\boldsymbol{\mu}}$ is neutral if and only if*

$$f(p) + f(1-p) = 2f(1/2), \quad \forall p \in [0,1]. \tag{8}$$

We further abuse notation and say that f is neutral if it satisfies (8). When both f and s are neutral, we know that f-NAWM satisfies neutrality by Lemma 4.5. We now give a precise characterization of when an f-NAWM satisfies weak budget balance, which is essentially when $s_0(f^{-1}(\cdot))$ is concave.

THEOREM 5.3. *The necessary and sufficient conditions for a differentiable f-NAWM to satisfy weak budget balance are respectively:*

(1) *A differentiable f-NAWM is weakly budget balanced if f and s are neutral and $s_0(f^{-1}(\cdot))$ is concave.*
(2) *If a differentiable f-NAWM is weakly budget balanced then $s_0(f^{-1}(\cdot))$ is concave.*

The proof relies on the following lemma, which gives a necessary condition for general NAWMs that is a partial converse of Theorem 4.6, in the sense that the inequality $\bar{p}_i \leq \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}$ is required to hold for only certain vectors.

LEMMA 5.4. *Let p, q be any two numbers in $[0,1]$. Consider a differentiable NAWM that is weakly budget balanced. Fix all the wagers to be equal to 1 and let $\bar{p}_i = \bar{p}(\mathbf{p}_{-i})$ for some function $\bar{p} : [0,1]^{n-1} \to [0,1]$. Let $\mathbf{1}_{n-2}$ be the vector of 1's in $n-2$ dimensions. Then*

$$\bar{p}(p, q\mathbf{1}_{n-2}) \leq \|(p, q\mathbf{1}_{n-2})\|_{s_0,\boldsymbol{\mu}}\ \text{, where } \mu_j = 1/(n-1) \text{ for all } j.$$

PROOF OF THEOREM 5.3. The first part follows easily from Theorem 4.6 and Lemma 2.6.

For the second part, first observe that applying Lemma 5.4 to an f-NAWM implies that for any p and q, letting $\boldsymbol{\mu} = (1/k, 1-1/k)$ for any integer $k \geq 2$ and $\boldsymbol{\mu}'$ be a k-dimensional vector with $\mu'_j = 1/k$ for all j, we have

$$\|(p,q)\|_{f,\boldsymbol{\mu}} = \|(p, q\mathbf{1}_{k-1})\|_{f,\boldsymbol{\mu}'} \leq \|(p, q\mathbf{1}_{k-1})\|_{s_0,\boldsymbol{\mu}'} = \|(p,q)\|_{s_0,\boldsymbol{\mu}}.$$

The above inequality gives that for $h(x) = s_0(f^{-1}(x))$, $\alpha h(x) + (1-\alpha)h(y) \leq h(\alpha x + (1-\alpha)y)$, for all x, y, and for all $\alpha = 1/k$ for some integer $k \geq 2$. To see this, for any x and y, let $p = f^{-1}(x)$ and $q = f^{-1}(y)$. Then,

$$\begin{aligned}
\|(p,q)\|_{s_0,(\alpha,1-\alpha)} &\geq \|(p,q)\|_{f,(\alpha,1-\alpha)} \\
\Leftrightarrow s_0^{-1}(\alpha s_0(p) + (1-\alpha)s_0(q)) &\geq f^{-1}(\alpha f(p) + (1-\alpha)f(q)) \\
\Leftrightarrow \alpha s_0(p) + (1-\alpha)s_0(q) &\leq s_0(f^{-1}(\alpha f(p) + (1-\alpha)f(q))) \\
\Leftrightarrow \alpha s_0(f^{-1}(x)) + (1-\alpha)s_0(f^{-1}(y)) &\leq s_0(f^{-1}(\alpha x + (1-\alpha)y)) \\
\Leftrightarrow \alpha h(x) + (1-\alpha)h(y) &\leq h(\alpha x + (1-\alpha)y),
\end{aligned}$$

where the inequality changes direction because s_0 is a decreasing function.

We need to show that this inequality holds for all $\alpha \in [0,1]$. We show this for a dense subset of $[0,1]$, and since h is continuous, it follows that it holds for any $\alpha \in [0,1]$. The dense set that we show it for is the set of all rational numbers where the denominator is a power of 2. This follows from the following recursive construction, where we show that if the above inequality holds for α_1 and α_2, then it also holds for $(\alpha_1+\alpha_2)/2$.

$$\begin{aligned}
h\left(\frac{\alpha_1+\alpha_2}{2}x + \left(1 - \frac{\alpha_1+\alpha_2}{2}\right)y\right) &= h\left(\frac{1}{2}\left(\alpha_1 x + (1-\alpha_1)y\right) + \frac{1}{2}\left(\alpha_2 x + (1-\alpha_2)y\right)\right) \\
&\geq \frac{1}{2}h\left(\alpha_1 x + (1-\alpha_1)y\right) + \frac{1}{2}h\left(\alpha_2 x + (1-\alpha_2)y\right) \\
&\geq \frac{\alpha_1}{2}h(x) + \frac{1-\alpha_1}{2}h(y) + \frac{\alpha_2}{2}h(x) + \frac{1-\alpha_2}{2}h(y) \\
&= \frac{\alpha_1+\alpha_2}{2}h(x) + \left(1 - \frac{\alpha_1+\alpha_2}{2}\right)h(y).
\end{aligned}$$

In the first inequality, we used $\alpha h(x) + (1-\alpha)h(y) \leq h(\alpha x + (1-\alpha)y)$,with $\alpha = 1/2$ and in the second inequality we used it with α_1 and α_2. This completes the proof. $\square$

5.1. Example f-NAWMs

Next, we give specific examples of f-NAWMs that are neutral and weakly budget balanced. The first follows easily from Lemma 3.2 and our original motivation.

LEMMA 5.5. *Let s be a neutral, differentiable, proper scoring rule and G be the corresponding convex function as in Theorem 2.2. Then the f-NAWM with $f = G'$ using scoring rule s is weakly budget balanced and neutral. Further, the surplus of the mechanism is the same for both outcomes.*

The class of f-NAWMs described in Lemma 5.5 is the most balanced in the sense that the surplus is the same no matter what the outcome is. We now present two other choices that are in that sense the most extreme mechanisms.

For any f defined on $[0, 1]$ that satisfies (8), its value on $[0, 1/2]$ completely determines its value on $(1/2, 1]$. Hence, we now define an operation that takes any continuous strictly monotone function defined on $[0, 1/2]$ and extends it to a continuous, strictly monotone function on $[0, 1]$ that satisfies (8). Given any function $h : [0, 1] \to \mathbf{R}$, define the *symmetrization* of the function, denoted $\mathrm{sym}(h)$, as

$$\mathrm{sym}(h)(p) = \begin{cases} h(p) & \text{if } p \in [0, 1/2], \\ 2h(1/2) - h(1-p) & \text{if } p \in (1/2, 1]. \end{cases} \tag{9}$$

It is easy to verify that for any function h defined on $[0, 1/2]$, the new function $\mathrm{sym}(h)$ is defined on $[0, 1]$ and satisfies (8). We will use the notation $\mathrm{sym}(p^2)$ to denote the symmetrization $\mathrm{sym}(h)$ of the function $h(p) = p^2$ and similarly for other common functions.

THEOREM 5.6. *Let s be a neutral, differentiable, proper scoring rule. The f-NAWM with $f = \mathrm{sym}(s_0)$ or $f = \mathrm{sym}(s_1)$ using scoring rule s is weakly budget balanced and neutral. When the outcome is x, the surplus of the mechanism with $f = \mathrm{sym}(s_0)$ is 0 when all of the predictions are closer to x than to $1 - x$; the surplus of the mechanism with $f = \mathrm{sym}(s_1)$ is 0 when all of the predictions are closer to $1 - x$ than to x.*

The profit of the f-NAWM with $f = \mathrm{sym}(s_0)$ has the following nice interpretation: if everyone predicted that 0 was more likely than 1 and 0 is the outcome (i.e. all agents make "correct" predictions), then the mechanism is exactly budget balanced and doesn't make a profit. This may be seen as being closer to the property of exact budget balanced than the f-NAWM with $f = G'$, which will always have a positive profit unless all predictions are the same. On the other hand, the f-NAWM with $f = \mathrm{sym}(s_1)$ obtains a profit in a less natural scenario, in that it makes a positive profit when all agents predicted 0 as more likely than 1 and the outcome is 0 (i.e. all agents make "correct" predictions), but makes zero profit when everyone predicted 0 as more likely than 1 and the outcome is 1 (i.e. all agents make "wrong" predictions)!

We next turn our attention to the Brier scoring rule, s^B, which is defined in (1) and satisfies neutrality. We give a whole range of functions f for which an f-NAWM using the Brier scoring rule is weakly budget balanced and neutral. This range is an interpolation between the two extremes, $\mathrm{sym}(s_0)$ and $\mathrm{sym}(s_1)$.

LEMMA 5.7. *If either $f = \mathrm{sym}(p^a)$ or $f = \mathrm{sym}((1-p)^a)$ for some constant $a \in [1, 2]$, then the f-NAWM with $s = s^B$ is weakly budget balanced and neutral.*

5.2. Sybilproofness

We now show that the class of neutral and weakly budget balanced f-NAWMs also satisfies sybilproofness.

THEOREM 5.8. *Any neutral and weakly budget balanced f-NAWM is sybilproof.*

Neutrality and weak budget balance of an f-NAWM mean that s and $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ are neutral (or an equivalent mechanism can be written with a neutral s) and $s_0(f^{-1}(\cdot))$ is concave. Our proof first shows that we only need to prove sybilproofness for the case of two agents, i and j. Then, using the neutrality of $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ and concavity of $s_0(f^{-1}(\cdot))$, we can show that agent i's expected net payoff is higher when he predicts p_i and wagers w_i than when he participates under any number $k > 1$ of identities and predicts p_{i_ℓ} and wagers w_{i_ℓ} for his identity ℓ, where p_{i_ℓ} and w_{i_ℓ} can be arbitrary predictions and wagers as long as $\sum_{\ell=1}^{k} w_{i_\ell} = w_i$.

6. CONCLUSION

We establish a method to construct wagering mechanisms that satisfy anonymity, individual rationality, incentive compatibility, no arbitrage, weak budget balance, neutrality, and sybilproofness, and provide structural characterizations for wagering mechanisms satisfying no arbitrage and some subsets of the other properties. These mechanisms allow the center to make a guaranteed profit from the disagreement of agents with immutable beliefs, without sacrificing major incentives properties. An intriguing future direction is to characterize all one-shot wagering mechanisms that satisfy all seven properties.

While we present our analysis in a setting for predicting binary random variables, some of our results naturally generalize to predicting finite discrete random variables. In particular, the "arbitrage interval" characterized in Theorem 3.3 easily generalizes to an arbitrage set of probability vectors, and this condition can be used to generalize Theorem 4.4 for NAWMs to satisfy weak budget balance.

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A. MISSING PROOFS

A.1. Missing Proofs in Section 2

PROOF OF COROLLARY 2.3. Since $s_x(p)$ is differentiable, by Theorem 2.2, $G(p)$ is twice differentiable. Then, $s_1'(p) = (1-p)G''(p)$ and $s_0'(p) = -pG''(p)$. Because s is strictly proper, $G(p)$ is strictly convex and $G''(p) > 0$. Hence, $s_1'(p) > 0$ for $p \in [0, 1)$ and $s_0'(p) < 0$ for $p \in (0, 1]$. Thus, $s_1(p)$ strictly increases and $s_0(p)$ strictly decreases with p. $\square$

PROOF OF LEMMA 2.5. Let $g(x) = af(x) + b$. For any particular x with $y = g(x)$, we have $f(x) = \frac{y-b}{a}$ and $g^{-1}(y) = x = f^{-1}(\frac{y-b}{a})$. Then

$$\begin{aligned}
\|\mathbf{p}\|_{g,\boldsymbol{\mu}} &= g^{-1}\left(\boldsymbol{\mu}\text{–avg}(g(\mathbf{p}))\right) \\
&= f^{-1}\left(\frac{\boldsymbol{\mu}\text{–avg}(g(\mathbf{p})) - b}{a}\right) \\
&= f^{-1}\left(\frac{\boldsymbol{\mu}\text{–avg}(af(\mathbf{p}) + b) - b}{a}\right) \\
&= f^{-1}\left(\frac{a \cdot \boldsymbol{\mu}\text{–avg}(f(\mathbf{p})) + b - b}{a}\right) \\
&= f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(\mathbf{p}))\right) \\
&= \|\mathbf{p}\|_{f,\boldsymbol{\mu}}.
\end{aligned}$$

Now let $g(x) = f(ax + b)$. For any particular x with $y = g(x)$, $ax + b = f^{-1}(y)$ and $\frac{f^{-1}(y)-b}{a} = x = g^{-1}(y)$. Then

$$\begin{aligned}
\|\mathbf{p}\|_{g,\boldsymbol{\mu}} &= g^{-1}\left(\boldsymbol{\mu}\text{–avg}(g(\mathbf{p}))\right) \\
&= \frac{f^{-1}\left(\boldsymbol{\mu}\text{–avg}(g(\mathbf{p}))\right) - b}{a} \\
f^{-1}\left(\boldsymbol{\mu}\text{–avg}(g(\mathbf{p}))\right) &= f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(a\mathbf{p} + b))\right) \\
&= \|a\mathbf{p} + b\|_{f,\boldsymbol{\mu}}.
\end{aligned}$$

$\square$

PROOF OF LEMMA 2.6. For any convex function h, by definition of convexity, we have that for any x and vector of weights $\boldsymbol{\mu}$,

$$h(\boldsymbol{\mu}\text{–avg}(\mathbf{x})) \leq \boldsymbol{\mu}\text{–avg}(h(\mathbf{x})).$$

First we show the $\Leftarrow$ direction. Assume that g is increasing and $h(x) = g(f^{-1}(x))$ is convex. Given some $\mathbf{p} \in [0, 1]^n$, let $x_i = f(p_i)$. Applying the convexity inequality above, we get that for any vector of weights $\boldsymbol{\mu}$,

$$\begin{aligned}
g\left(f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(\mathbf{p}))\right)\right) &\leq \boldsymbol{\mu}\text{–avg}\left(g\left(f^{-1}(f(\mathbf{p}))\right)\right) = \boldsymbol{\mu}\text{–avg}(g(\mathbf{p})) = g(g^{-1}\left(\boldsymbol{\mu}\text{–avg}(g(\mathbf{p}))\right)) \\
\Leftrightarrow f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(\mathbf{p}))\right) &\leq g^{-1}\left(\boldsymbol{\mu}\text{–avg}(g(\mathbf{p}))\right),
\end{aligned}$$

since g is strictly increasing.

If g is decreasing and h is concave, then $-g$ is increasing and $-h$ is convex. Further, by Lemma 2.5, $\|\mathbf{p}\|_{g,\boldsymbol{\mu}} = \|\mathbf{p}\|_{-g,\boldsymbol{\mu}}$, so we may use the first part.

Now for the $\Rightarrow$ direction, once again assume that g is increasing, and assume that $\|\mathbf{p}\|_{f,\boldsymbol{\mu}} \leq \|\mathbf{p}\|_{g,\boldsymbol{\mu}}\ \forall\ \boldsymbol{\mu}$. We wish to show that $h(x) = g(f^{-1}(x))$ is convex. Let $x = f(p)$ and

$y = f(q)$ for any $p, q \in [0,1]$. For any $\alpha \in [0,1]$, we want to show that

$$\begin{aligned} \alpha h(x) + (1-\alpha) h(y) &\geq h(\alpha x + (1-\alpha) y) \\ \Leftrightarrow \alpha g(f^{-1}(x)) + (1-\alpha) g(f^{-1}(y)) &\geq g(f^{-1}(\alpha x + (1-\alpha) y)) \\ \Leftrightarrow \alpha g(p) + (1-\alpha) g(q) &\geq g(f^{-1}(\alpha f(p) + (1-\alpha) f(q))) \\ \Leftrightarrow g^{-1}(\alpha g(p) + (1-\alpha) g(q)) &\geq f^{-1}(\alpha f(p) + (1-\alpha) f(q)) \\ \Leftrightarrow \|(p,q)\|_{g,(\alpha,1-\alpha)} &\geq \|(p,q)\|_{f,(\alpha,1-\alpha)} \end{aligned}$$

The case where g is decreasing can be handled similarly, except that the inequality is reversed when applying g^{-1} to both sides. $\square$

A.2. Missing Proofs in Section 4

PROOF OF THEOREM 4.4. The proof follows from the observation in (6) and the budget balance of WSWMs. Because $\bar{p}_i \in [\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}, \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}]$, according to Theorem 3.3, when not all elements of $\mathbf{p}_{-i}$ are the same,

$$\Pi_i^{\mathrm{WS}}(\bar{p}_i, \mathbf{p}_{-i}, \mathbf{w}, x) \geq 0,$$

for all $x \in \{0,1\}$. When all elements of $\mathbf{p}_{-i}$ are the same and equal to p, we have $\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}} = \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}} = p$. Hence, $\bar{p}_i$ must also equal to p and in this case $\Pi_i^{\mathrm{WS}}(\bar{p}_i, \mathbf{p}_{-i}, \mathbf{w}, x) = 0$ for all $x \in \{0,1\}$. Since the WSWM is budget balanced, by (6),

$$\sum_{i \in \mathcal{N}} \Pi_i^{\mathrm{NA}}(\mathbf{p}, \mathbf{w}, x) = \sum_{i \in \mathcal{N}} \Pi_i^{\mathrm{WS}}(\mathbf{p}, \mathbf{w}, x) - \sum_{i \in \mathcal{N}} \Pi_i^{\mathrm{WS}}(\bar{p}_i, \mathbf{p}_{-i}, \mathbf{w}, x) = -\sum_{i \in \mathcal{N}} \Pi_i^{\mathrm{WS}}(\bar{p}_i, \mathbf{p}_{-i}, \mathbf{w}, x) \leq 0.$$

$\square$

PROOF OF THEOREM 4.6. Because the NAWM satisfies neutrality, both s and $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ are neutral. Since s is neutral, $s_1(p) = s_0(1-p)$. Using this in Lemma2.5, we get that

$$\|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}} = 1 - \|\mathbf{1} - \mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}.$$

If $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) \leq \|\mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}}$ for all $\mathbf{p}$, and $\mathbf{w}$ and $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ is neutral, then

$$\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i}) = 1 - \bar{p}(\mathbf{1} - \mathbf{p}_{-i}, \mathbf{w}_{-i}) \geq 1 - \|\mathbf{1} - \mathbf{p}_{-i}\|_{s_0,\boldsymbol{\mu}} = \|\mathbf{p}_{-i}\|_{s_1,\boldsymbol{\mu}}.$$

By Theorem 4.4, the NAWM satisfies weak budget balance. $\square$

A.3. Missing Proofs in Section 5

Two lemmas will be used in the proofs given in this section. We state them first.

LEMMA A.1. *Let f and g be twice differentiable, strictly monotone functions.*

$$\frac{f''}{f'} \leq \frac{g''}{g'} \Leftrightarrow \begin{cases} g(f^{-1}(\cdot)) \text{ is convex} & \text{if } g \text{ is increasing} \\ g(f^{-1}(\cdot)) \text{ is concave} & \text{if } g \text{ is decreasing.} \end{cases}$$

PROOF. Let $h(x) = g(f^{-1}(x))$. Since f and g are twice differentiable, so is h, and

$$h'(x) = g'(f^{-1}(x))(f^{-1})'(x) \text{ and}$$

$$h''(x) = g'(f^{-1}(x))(f^{-1})''(x) + ((f^{-1})'(x))^2 g''(f^{-1}(x)).$$

Let $y = f^{-1}(x)$. Then

$$(f^{-1})'(x) = \frac{1}{f'(y)} \text{ and } (f^{-1})''(x) = \frac{-f''(y)}{f'(y)^3}.$$

Substituting these, we get

$$h''(x) = \frac{-g'(y)f''(y)}{f'(y)^3} + \frac{g''(y)}{f'(y)^2}.$$

It is easy to check that

$$\frac{f''}{f'} \leq \frac{g''}{g'} \Leftrightarrow \begin{cases} h''(x) \geq 0 & \text{if } g'(y) \geq 0 \\ h''(x) \leq 0 & \text{if } g'(y) \leq 0. \end{cases}$$

□

LEMMA A.2. *Fix a function* $h : [0,1] \to \mathbf{R}$*, and let* $f = \mathrm{sym}(h)$*. Then for* $p \in (\frac{1}{2}, 1]$*,* $f'(p) = h'(1-p)$ *and* $f''(p) = -h''(1-p)$*.*

The proof of Lemma A.2 is immediate using the definition of symmetrization in (9).

PROOF OF LEMMA 5.2 . If f is strictly monotone and continuous, its inverse f^{-1} is well defined. If f satisfies (8), then for any p, letting $y = f(p)$ gives us

$$f^{-1}\left(2f(\tfrac{1}{2}) - y\right) = 1 - f^{-1}(y).$$

Using this and (8), we have

$$\begin{aligned} \|\mathbf{1} - \mathbf{p}\|_{f,\boldsymbol{\mu}} &= f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(\mathbf{1} - \mathbf{p}))\right) \\ &= f^{-1}\left(2f(\tfrac{1}{2}) - \boldsymbol{\mu}\text{–avg}(f(\mathbf{p}))\right) \\ &= 1 - f^{-1}\left(\boldsymbol{\mu}\text{–avg}(f(\mathbf{p}))\right) \\ &= 1 - \|\mathbf{p}\|_{f,\boldsymbol{\mu}}. \end{aligned}$$

For the other direction, assume that $\|\mathbf{1} - \mathbf{p}\|_{f,\boldsymbol{\mu}} = 1 - \|\mathbf{p}\|_{f,\boldsymbol{\mu}}$. Let $\mathbf{p} = (p, 1-p)$ and $\mu = (1/2, 1/2)$. We have

$$\begin{aligned} f^{-1}\left(\frac{f(p) + f(1-p)}{2}\right) &= 1 - f^{-1}\left(\frac{f(p) + f(1-p)}{2}\right) \\ \Rightarrow 2f^{-1}\left(\frac{f(p) + f(1-p)}{2}\right) &= 1 \\ \Rightarrow f(p) + f(1-p) &= 2f\left(\tfrac{1}{2}\right). \end{aligned}$$

□

PROOF OF LEMMA 5.4. Suppose that the prediction vector $\mathbf{p}$ is $(p, q\mathbf{1}_{n-1})$, i.e., participant 1 predicts p and everyone else predicts q. Then using anonymity and weak budget balance, it can be argued that $\bar{p}_1 = q$, and we have $\bar{p}_j = \bar{p}(p, q\mathbf{1}_{n-2})$ for all $j \neq 1$. Now using the weak budget balance condition, we get that

$$\begin{aligned} s_0(p) - s_0(q) + (n-1)s_0(q) - (n-1)s_0(\bar{p}(p, q\mathbf{1}_{n-2})) &\leq 0 \\ \Rightarrow s_0(p) + (n-2)s_0(q) &\leq (n-1)s_0(\bar{p}(p, q\mathbf{1}_{n-2})) \\ \Rightarrow \frac{1}{n-1}(s_0(p) + (n-2)s_0(q)) &\leq s_0(\bar{p}(p, q\mathbf{1}_{n-2})) \\ \Rightarrow s_0^{-1}\left(\frac{1}{n-1}(s_0(p) + (n-2)s_0(q))\right) &\geq \bar{p}(p, q\mathbf{1}_{n-2}) \\ \Rightarrow \|(p, q\mathbf{1}_{n-2})\|_{s_0,\boldsymbol{\mu}} &\geq \bar{p}(p, q\mathbf{1}_{n-2}). \end{aligned}$$

Most of the inequalities are self-explanatory. The reversal of the inequality is because s_0 is decreasing. □

PROOF OF LEMMA 5.5. That this mechanism satisfies weak budget balance follows essentially from Theorem 4.4 using Lemma 3.2 and Theorem 3.3. The fact that the surplus is independent of the outcome follows from the fact that the arbitrage profit is independent of the outcome in Lemma 3.2.

By Lemma 4.5 and Lemma 5.2, to show neutrality, we must simply show that G' is neutral assuming neutrality of s. From Theorem 2.2,

$$G(p) = ps_1(p) + (1-p)s_0(p) = ps_0(1-p) + (1-p)s_0(p).$$

$$G'(p) = s_0(1-p) - ps_0'(1-p) - s_0(p) + (1-p)s_0'(p).$$

$$G'(1-p) = s_0(p) - (1-p)s_0'(p) - s_0(1-p) + ps_0'(1-p).$$

Adding, we get that $G'(p) + G'(1-p) = 0 = G'(1/2)$ as desired. □

PROOF OF LEMMA 5.7. It is easy to check that s^B is neutral. Therefore from Lemma A.1 and Theorem 5.3, it is sufficient to show that $f''/f' \le s_0''/s_0'$. We first consider $f = \mathrm{sym}(p^a)$.

Case 1: $p \in [0, 1/2]$

We calculate, $s_0'(p) = -2p$, $s_0''(p) = -2$ and $s_0''/s_0' = 1/p$. Similarly, $f'(p) = ap^{a-1}$, $f''(p) = a(a-1)p^{a-2}$ and $f''/f' = (a-1)/p$. Since $a \le 2$, we have that $f''/f' \le s_0''/s_0'$.

Case 2: $p \in (1/2, 1]$

s_0''/s_0' is, as before, $1/p \ge 0$. We now use Lemma A.2 to calculate $f'(p) = a(1-p)^{a-1}$, $f''(p) = -a(a-1)(1-p)^{a-2}$ and $f''/f' = -(a-1)/(1-p) \le 0$ since $a \ge 1$, so once again we have that $f''/f' \le s_0''/s_0'$.

Note that this is tight. If $a > 2$, then Case 1 fails. If $a < 1$, then Case 2 fails since $-(a-1)/(1-p)$ would be greater than $1/p$ for p sufficiently close to 1, for any given $a < 1$.

We now consider $f = \mathrm{sym}((1-p)^a)$.

Case 1: $p \in [0, 1/2]$

$f'(p) = -a(1-p)^{a-1}$, $f''(p) = a(a-1)(1-p)^{a-2}$ and $f''/f' = -(a-1)/(1-p) \le 0 \le s_0''/s_0'$.

Case 2: $p \in (1/2, 1]$

Using Lemma A.2, $f'(p) = -ap^{a-1}$, $f''(p) = -a(a-1)p^{a-2}$ and $f''/f' = (a-1)/p$, so once again since $a \le 2$, we have that $f''/f' \le s_0''/s_0'$.

□

PROOF OF THEOREM 5.6. For weak budget balance, from Theorem 5.3 it is sufficient to show that $s_0(f^{-1})$ is concave. First consider $f = \mathrm{sym}(s_0)$. Since $f = s_0$ on $[0, 1/2]$, it follows that $s_0(f^{-1})$ is identity on $[0, 1/2]$. Therefore, the non-trivial part is to show that $s_0(f^{-1})$ is concave on $(1/2, 1]$, where $f(p) = 2s_0(1/2) - s_0(1-p)$.

Using Lemma 2.5 we get that when $\mathbf{p} \in (1/2, 1]^n$, $\|\mathbf{p}\|_{f,\boldsymbol{\mu}} = 1 - \|\mathbf{p}\|_{s_0,\boldsymbol{\mu}}$. It is easy to see that $\|\mathbf{p}\|_{s_0,\boldsymbol{\mu}} > 1/2$ when $\mathbf{p} \in (1/2, 1]^n$, and therefore $1 - \|\mathbf{p}\|_{s_0,\boldsymbol{\mu}} \le \|\mathbf{p}\|_{s_0,\boldsymbol{\mu}}$. Now we use Lemma 2.6 in the $\Rightarrow$ direction to conclude that $s_0(f^{-1})$ is concave on $(1/2, 1]$.

We have shown that $s_0(f^{-1})$ is concave on $[0, 1/2]$ and $(1/2, 1]$ separately. But then f is differentiable at $1/2$, which implies so is $s_0(f^{-1})$, and the derivative is continuous at $1/2$. Therefore $s_0(f^{-1})$ is concave on the entire $[0, 1]$.

One can easily verify that when all of the predictions are at most $1/2$ and the outcome is 0, $\|\mathbf{p}\|_{f,\boldsymbol{\mu}} = \|\mathbf{p}\|_{s_0,\boldsymbol{\mu}}$, and the sum of payments to the agents is 0. Since this is a neutral NAWM, the same is true when predictions are at least $1/2$ and the outcome is 1.

The proof for $f = \mathrm{sym}(s_1)$ is similar. □

PROOF OF THEOREM 5.8. The neutrality and weak budget balance of a f-NAWM imply that s and $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ are neutral and $s_0(f^{-1}(\cdot))$ is concave. Let $s(r, q) =$

$qs_1(r) + (1-q)s_0(r)$ be the expected score for prediction r under belief q. We first prove a condition that we will use later, that for any $\mathbf{p}$, q, and $\boldsymbol{\mu}$,

$$s(\|\mathbf{p}\|_{f,\boldsymbol{\mu}}, q) \geq \sum_i \mu_i s(p_i, q). \tag{10}$$

We have

$$\begin{aligned}
s(\|\mathbf{p}\|_{f,\boldsymbol{\mu}}, q) &= q\, s_1(\|\mathbf{p}\|_{f,\boldsymbol{\mu}}) + (1-q)s_0(\|\mathbf{p}\|_{f,\boldsymbol{\mu}}) \\
&= q\, s_0(1 - \|\mathbf{p}\|_{f,\boldsymbol{\mu}}) + (1-q)s_0(\|\mathbf{p}\|_{f,\boldsymbol{\mu}}) \\
&= q\, s_0(\|\mathbf{1} - \mathbf{p}\|_{f,\boldsymbol{\mu}}) + (1-q)s_0(\|\mathbf{p}\|_{f,\boldsymbol{\mu}}) \\
&= q\, s_0(f^{-1}(\sum_i \mu_i f(1-p_i))) + (1-q)s_0(f^{-1}(\sum_i \mu_i f(p_i))) \\
&\geq q \sum_i \mu_i s_0(f^{-1}(f(1-p_i))) + (1-q)\sum_i \mu_i s_0(f^{-1}(f(p_i))) \\
&= q \sum_i \mu_i s_0(1-p_i) + (1-q)\sum_i \mu_i s_0(p_i) \\
&= q \sum_i \mu_i s_1(p_i) + (1-q)\sum_i \mu_i s_0(p_i) \\
&= \sum_i \mu_i s(p_i, q).
\end{aligned}$$

The second equality is due to the neutrality of s. The third equality is because the neutrality of $\bar{p}(\mathbf{p}_{-i}, \mathbf{w}_{-i})$ implies that $1 - \|\mathbf{p}\|_{f,\boldsymbol{\mu}} = \|\mathbf{1} - \mathbf{p}\|_{f,\boldsymbol{\mu}}$. The inequality is due to the concavity of $s_0(f^{-1}(\cdot))$.

Now we are ready to prove sybilproofness. We first show that it is sufficient to prove sybilproofness for two-agent wagering. In other words, we show that if an agent would not want to create false identities when wagering against any single agent, she would also not want to create false identities when wagering against any group of agents.

From the definition of a NAWM, it is clear that when agent i participates using her own identity, she receives the same payoff she would receive if she were playing against a single other agent with prediction and wager pair $(\bar{p}_i, W_{\mathcal{N}\setminus\{i\}})$. Suppose agent i participates under K identities, $i_1 \ldots i_K$, and they make predictions and wagers $(p_{i_1}, w_{i_1}) \ldots (p_{i_k}, w_{i_K})$ respectively, where $\sum_{k=1}^{K} w_{i_k} = w_i$ and $w_{i_k} \geq 0$, $\forall k$. Then $\bar{p}_{i_k}$ for identity i_k satisfies the following condition:

$$\begin{aligned}
\bar{p}_{i_k} &= f^{-1}\left(\sum_{j\in\mathcal{N}\cup\{i_1,\ldots,i_K\}\setminus\{i,i_k\}} \frac{w_j}{W_{\mathcal{N}\setminus\{i\}} + w_i - w_{i_k}} f(p_j)\right) \\
&= f^{-1}\left(\frac{W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}\setminus\{i\}} + w_i - w_{i_k}} \sum_{j\in\mathcal{N}\setminus\{i\}} \frac{w_j}{W_{\mathcal{N}\setminus\{i\}}} f(p_j) + \sum_{j\in\{1,\ldots K\}\setminus\{k\}} \frac{w_{i_j}}{W_{\mathcal{N}\setminus\{i\}} + w_i - w_{i_k}} f(p_{i_j})\right) \\
&= f^{-1}\left(\frac{W_{\mathcal{N}\setminus\{i\}}}{W_{\mathcal{N}\setminus\{i\}} + w_i - w_{i_k}} f(\bar{p}_i) + \sum_{j\in\{1,\ldots K\}\setminus\{k\}} \frac{w_{i_j}}{W_{\mathcal{N}\setminus\{i\}} + w_i - w_{i_k}} f(p_{i_j})\right).
\end{aligned}$$

For each identity, the reference report $\bar{p}_{i_k}$ is calculated in the same way it would be if the identity were playing against a single agent with prediction and wager pair $(\bar{p}_i, W_{\mathcal{N}\setminus\{i\}})$ plus the other identities of agent i. Thus, to prove sybilproofness, we only need to consider the case where player i plays against a single other player.

Now consider player i with prediction and wager (p_i, w_i) and another player with prediction and wager (p, w). If player i participates using her true identity, her expected net payoff is

$$E_{x\sim p_i}[\Pi_i((p_i, p), (w_i, w), x)] = \frac{w}{w + w_i} w_i(s(p_i, p_i) - s(p, p_i)). \tag{11}$$

If agent i participates under identities $i_1, \ldots, i_K$ and they make predictions and wagers $(p_{i_1}, w_{i_1}), \ldots, (p_{i_K}, w_{i_K})$ respectively, where $w_{i_k} \geq 0$ and $\sum_{k=1}^{K} w_{i_k} = w_i$, then the expected payoff of identity k is

$$E_{x\sim p_i}[\Pi_{i_k}((p_{i_1}, \ldots, p_{i_K}, p), (w_{i_1}, \ldots, w_{i_K}, w), x)] = \frac{w + w_i - w_{i_k}}{w + w_i} w_{i_k}(s(p_{i_k}, p_i) - s(\bar{p}_{i_k}, p_i)). \tag{12}$$

To show sybilproofness, we'll show that for any (p, w), K, (p_i, w_i), and $(p_{i_1}, w_{i_1}), \ldots, (p_{i_K}, w_{i_K})$, the right hand side of (11) is at least as big as the right hand side of (12).

For identity i_k, we know that

$$\bar{p}_{i_k} = f^{-1}\left(\frac{w}{w + w_i - w_{i_k}} f(p) + \sum_{j\in\{1,\ldots K\}\setminus\{k\}} \frac{w_{i_j}}{w + w_i - w_{i_k}} f(p_{i_j})\right).$$

The expected total payoff of the K identities is

$$\begin{aligned}
&\sum_{k=1}^{K} \frac{w + w_i - w_{i_k}}{w + w_i} w_{i_k}(s(p_{i_k}, p_i) - s(\bar{p}_{i_k}, p_i)) \\
&\leq \sum_{k=1}^{K} \frac{w + w_i - w_{i_k}}{w + w_i} w_{i_k}\left(s(p_{i_k}, p_i) - \frac{w}{w + w_i - w_{i_k}} s(p, p_i) - \sum_{j\in\{1,\ldots K\}\setminus\{k\}} \frac{w_{i_j}}{w + w_i - w_{i_k}} s(p_{i_j}, p_i)\right) \\
&= \sum_{k=1}^{K} \frac{w + w_i - w_{i_k}}{w + w_i} w_{i_k}\left(\frac{w + w_i}{w + w_i - w_{i_k}} s(p_{i_k}, p_i) - \frac{w}{w + w_i - w_{i_k}} s(p, p_i) - \sum_{j\in\{1,\ldots K\}} \frac{w_{i_j}}{w + w_i - w_{i_k}} s(p_{i_j}, p_i)\right) \\
&= \sum_{k=1}^{K} w_{i_k} s(p_{i_k}, p_) - \frac{w}{w + w_i} w_{i_k} s(p, p_i) - \frac{w_{i_k}}{w + w_i} \sum_{j=1}^{K} w_{i_j} s(p_{i_j}, p_i) \\
&= \frac{w}{w + w_i} w_i \left(\sum_{k=1}^{K} \frac{w_{i_k}}{w_i} s(p_{i_k}, p_i) - s(p, p_i)\right) \\
&\leq \frac{w}{w + w_i} w_i \left(s(f^{-1}(\sum_{k=1}^{K} \frac{w_{i_k}}{w_i} f(p_{i_k})), p_i) - s(p, p_i)\right) \\
&\leq \frac{w}{w + w_i} w_i \left(s(p_i, p_i) - s(p, p_i)\right)
\end{aligned}$$

which matches the right hand side of (12), as desired. The first two inequalities are due to (10). The last inequality is due to the fact that s is a proper scoring rule. □