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# Incentivizing Truthful Reports – Scoring Rules

Consider a rational agent with linear utility for cash.

Suppose there are $n$ mutually exclusive and exhaustive states of the world $\Omega = \{\omega_1, \ldots, \omega_n\}$

e.g., $n$ possible winners of the Republican primary

How can we motivate this agent to tell us his beliefs about the likelihood of each state?

$p_i$ = player's subjective belief about the probability state $\omega_i$ will occur

Scoring rules $S_i(\vec{r})$ = payment made to agent who reports distribution $\vec{r}$ if outcome is $\omega_i$

Agent's expected payoff is $\sum_{i=1}^{n} p_i S_i(\vec{r})$

Want to design scoring rules such that $\forall$ distributions $\vec{p}$,

$$\vec{p} = \underset{\vec{r}}{\operatorname{argmax}} \sum_{i=1}^{n} p_i s_i(\vec{r})$$

← if agent is rational with linear utility for money, they will report truthfully

such a scoring rule is called proper,

and strictly proper if $\vec{p}$ is the unique maximizer

Example: Quadratic Scoring Rule

$$S_i(\vec{r}) = a_i + b\left(r_i - \frac{\sum_j r_j^2}{2}\right)$$

Related to squared loss

$$\text{Squared loss of prediction} = -(1-r_i)^2 - \sum_{j \neq i} (0 - r_j)^2$$

$$= -1 + 2r_i - r_i^2 - \sum_{j \neq i} r_j^2$$

$$= \underbrace{-1}_{} + \underbrace{2r_i - \sum_j r_j^2}_{}$$

Can replace by any constant, even one that depends on i, without changing incentives

can scale by any factor without changing incentives (as long as the scaling factor doesn't depend on i)

Can verify this is incentive compatible by looking at the gradient of the Lagrangian of the expected payoff (Try this!)

Example: <u>Logarithmic Scoring Rule</u>

$$S_i(\vec{r}) = a_i + b \log(r_i)$$

Need to make $a_i$ big enough that agent won't lose money by participating

Related to <u>log loss</u>

Note that in order to use a scoring rule, it is necessary to have a sponsor who is willing to pay for information

Why choose a particular scoring rule?

- Quadratic has some nice symmetry properties
- Logarithmic is the only strictly proper scoring rule for which your score for outcome i depends only on $r_i$ and not $r_j$ for $j \neq i$

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# Hanson's Paper

Goal is to design a mechanism to incentivize multiple agents to share their beliefs, and a way to aggregate these beliefs into a unified prediction

* could use one scoring rule per agent, but it's unclear how to combine predictions
* could use standard information markets with stock market style trading, but there are several problems:
  - thin market, not enough traders (as in intrade's less popular markets)
  - "no trade theorems"

Solution: Market scoring rule

a "sequentially shared scoring rule"

- market maintains a vector of predictions $\vec{r}^t$
- if a trader changes the vector from $\vec{r}^t$ to $\vec{r}^{t+1}$ and outcome is $\omega_i$, trader gets

$$S_i(\vec{r}^{t+1}) - S_i(\vec{r}^t)$$

$$\text{Expected payoff} = \sum_{i=1}^{n} p_i \left( S_i(\vec{r}^{t+1}) - S_i(\vec{r}^t) \right)$$

$$= \underbrace{\sum_{i=1}^{n} p_i S_i(\vec{r}^{t+1})}_{\text{maximized by setting } \vec{r}^{t+1} = \vec{p}} - \underbrace{\sum_{i=1}^{n} p_i S_i(\vec{r}^t)}_{\text{doesn't depend on } \vec{r}^{t+1}}$$

$\longrightarrow$ will lead to truthful reports if
- agent will only trade once and will do it now (agent is "myopic")
- agent wants to maximize expect payoff

So we need rational myopic agents (plus even more assumptions if we want to get into aggregation...)
Also need to assume we can eventually learn the final outcome + organizer is willing to take a loss

Some nice properties:

- can bound the worst case loss of the market organizer $\longrightarrow$ organizer just needs to pay off the last trader

$$\max_i \; s_i(\vec{I}_i) - s_i(\vec{r}^{\,0})$$

Example: logarithmic market scoring rule with $\vec{r}^{\,0}$ uniform

$$\text{worst case loss} = b\log 1 - b\log\tfrac{1}{n} = b\log n$$

- can in principle handle large outcome spaces (though the details in this paper are vague)

- can view as an inventory-based market (though the details are really vague)

## Inventory-Baset markets w/ Cost Functions

$\Omega = \{\omega_1, \dots, \omega_n\}$

Suppose we want a market that sells securities of the form "\$1 if and only if state $\omega_i$"

Let $\vec{q}$ denote the number of securities that have been purchased

$q_i$ = # of securities on state $i$ purchased
(can be negative for sales)

c) A <u>cost function based market</u> allows purchases/sales of securities priced according to a potential function called a cost function, denoted $C$

Suppose I want to move quantity vector from $\vec{q}$ to $\vec{q}'$ (i.e., buy $\vec{q}_i' - \vec{q}_i$ of each contract $i$)

Then I must pay $C(\vec{q}') - C(\vec{q})$

price per share of an infinitesimally small share $= p_i = \dfrac{\partial C(\vec{q})}{\partial q_i}$ $\quad \vec{p} = \nabla C(\vec{q})$

<u>Claim</u>: Let $C(\vec{q}) = b \log\left(\sum_j e^{q_j/b}\right)$.

For all $\vec{p}, \vec{p}', \vec{q}, \vec{q}'$, such that

$\nabla C(\vec{q}) = \vec{p}$ and $\nabla C(\vec{q}') = \vec{p}'$

For all $i \in \{1, \dots, n\}$

Payoff for changing $\vec{p} \to \vec{p}'$ in LMSR for outcome $\omega_i$ = Payoff for changing $\vec{q} \to \vec{q}'$ in the CFBM for outcome $\omega_i$

$\Longrightarrow$ These two markets are equivalent

<u>Proof</u>: Through some basic calculus, you can verify that $p_i = \dfrac{e^{q_i/b}}{\sum_j e^{q_j/b}}$ $\quad p_i' = \dfrac{e^{q_i'/b}}{\sum_j e^{q_j'/b}} \ \forall i$

same price function given in Hanson's paper with no explanation c)

CFBM payoff $= q_i' - q_i - (C(\vec{q}') - C(\vec{q}))$

MSR payoff $= S_i(\vec{p}') - S_i(\vec{p})$

$$= b \log p_i' - b \log p_i$$

$$= b \log\left(\frac{e^{q_i'/b}}{\sum_j e^{q_j'/b}}\right) - b \log\left(\frac{e^{q_i/b}}{\sum_j e^{q_j/b}}\right)$$

$$= \underbrace{b \log e^{q_i'/b}}_{q_i'} - \underbrace{b \log \sum_j e^{q_j'/b}}_{C(\vec{q}')} - \underbrace{b \log e^{q_i/b}}_{q_i} + \underbrace{b \log \sum_j e^{q_j/b}}_{C(\vec{q})}$$

So we can implement the LMSR using this inventory-based representation (Is this better for UI reasons?)

LMSR is not special here – in general, MSRs can be viewed this way