Game Theory Intro Part 2

last time: utility, rational agents, games

Prisoner's Dilemma:

	C	D
C	-1, -1	-5, 0
D	0, -5	-4, -4

D is dominant strategy

Coordination Game:

	M	B
M	2, 1	0, 0
B	0, 0	1, 2

(M, M) and (B, B) are "stable"

We will now define the notion of a game more formally

<u>Def</u>: A finite, n-player <u>normal-form game</u> is a tuple $(N, A, \vec{u})$ where

- N is a finite set of n players, indexed by $i \in \{1, \dots, n\}$
- $A = A_1 \times \dots \times A_n$, where A_i is a finite set of actions available to player i
- $\vec{u} = \langle u_1, \dots, u_n \rangle$, where $u_i : A \to \mathbb{R}$ is a real-valued utility function for player i

Assumes all players choose actions at the same time

Terminology: Each vector $\vec{a} = \langle a_1, \dots, a_n \rangle \in A$ is an <u>action profile</u>

Another example: Rock, Paper, Scissors, a <u>zero-sum game</u>

	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

Is there a dominant strategy?

Is there a "stable" pair of actions?

Need a more general solution concept

a) $\boxed{\text{mixed strategies}}$ – allow uncertainty / randomization

Let S_i = set of all probability distributions over A_i

Each $s_i \in S_i$ is a mixed strategy, where $s_i(a_i)$ denotes the probability player i will choose action a_i

Payoffs determined by calculating expected utility

$$u_i(\vec{s}) = \sum_{\vec{a} \in A} u_i(\vec{a}) \prod_{j=1}^{n} s_j(a_j)$$

For example, in the coordination game, suppose

$$S_1(M) = \tfrac{1}{3} \quad S_1(B) = \tfrac{2}{3} \quad S_2(m) = \tfrac{1}{2} \quad S_2(B) = \tfrac{1}{2}$$

then $u_1(\vec{s}) = \frac{1}{6} \cdot 2 + \frac{1}{3} \cdot 0 + \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 1 = \frac{2}{3}$

Can work towards a more general def of "stable" state ...

$\boxed{\text{Best response}}$ – Let $\vec{s}$ be a "strategy profile" or set of strategies for all agents

Let $\vec{s}_{-i}$ denote the strategies of everyone except i

Def: A strategy $s_i^* \in S_i$ is a best response to the strategy profile s_{-i} if $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_i \in S_i$.

Not necessarily unique.

Important: If a best response puts positive probability on 2 or more actions, the agent must be indifferent between these actions (Why?)

$\boxed{\text{Nash equilibrium}}$ –

Def: A strategy profile $\vec{s}$ is a Nash equilibrium if, for all agents i, s_i is a best response to s_{-i}.

Can be weak or strong.

③ Examples:

Prisoner's Dilemma — D is a dominant strategy for both players → only one Nash equilibrium

Coordination Game — (m,B) and (B,m) are pure strategy Nash equilibria

Are there more?

Can use the indifference property

Suppose at equilibrium player 2 puts probability p on m and $(1-p)$ on B — call this S_2

For player 1 to be indifferent, need

$$u_1(m, s_2) = u_1(B, s_2)$$

$$2p + 0 = 0 + (1-p)$$

$$3p = 1 \qquad p = \tfrac{1}{3}$$

For player 2 to be indifferent assuming player 1 puts q on m and $1-q$ on B, need

$$q = 2(1-q) \qquad 3q = 2 \qquad q = \tfrac{2}{3}$$

So we have an equilibrium where Player 1 puts weight $\frac{2}{3}$ on m & player 2 puts weight $\frac{1}{3}$ on m

$$u_1(\vec{s}) = \frac{2}{9}\cdot 2 + \frac{2}{9}\cdot 1 = \frac{2}{3} = u_2(\vec{s})$$

→ lower expected utility than the other equilibria

Note on calculating equilibria: Easy in cases like this where we know/can guess the support of the strategies, but computationally hard in general

4 Rock, Paper, Scissors — Suppose player 2 uses probabilities p_r, p_p, p_s

Need $-p_p + p_s = p_r - p_s = -p_r + p_p$

Work out for yourself that the Nash equilibrium occurs when both players put weight $1/3$ on every action

This is the unique equilibrium

<u>Theorem</u>: (Nash, 51) Every game with a finite number of players and actions has at least one equilibrium.

<u>Other solution concepts</u> (mention if there is lots of time)

Correlated equilibria — allows shared randomness

Example: coordination game

prob $1/2$ on (M, M), $1/2$ on (B, B)

Players must not have incentive to deviate given the signal they receive

ε-nash — nobody gains more than ε by deviating

Can be useful for computational purposes

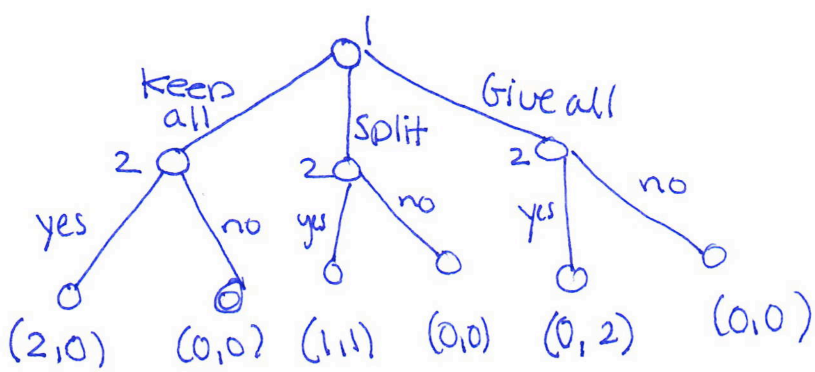
5) Extensive form games – allow explicit consideration of the sequence in which actions are taken

Sharing game example –

I give \$2 to player 1

Player 1 proposes an allocation – keep all, split w/ player 2, or give all to player 2

Player 2 then chooses whether to accept this allocation or return the money



Could write this as a normal form game with expanded action spaces

$A_1 = \{K, S, G\}$

$A_2 = \{yyy, yyn, yny, \dots, nnn\}$

24 outcomes vs. 6 in the extensive form

Definition of Nash can be imported directly – requires specifying an action for <u>every</u> node for every player

Example Nash equilibrium:

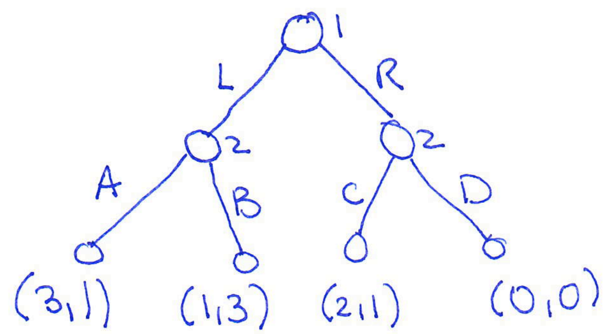
Player 1 chooses split

player 2 chooses (no, yes, yes)

Neither can benefit by deviating

Theorem: Every finite, perfect-information game in extensive form has a pure-strategy Nash equilibrium.

There is some weirdness about this natural extension of Nash equilibrium



Nash equilibrium:

player 1 chooses L

player 2 chooses (B, D)

Why is this strange? Is it really stable?

Subgame perfect Equilibrium The subgame perfect equilibria of a game G are all strategy profiles $\vec{s}$ such that for any subgame G' of G, the restriction of $\vec{s}$ to G' is a NE of G'.

Rules out "noncredible threats"

SPE:

Player 1 chooses R

player 2 chooses (B, C)

Can generally be found using backwards induction