CS 112: Modeling Uncertainty in Information Systems

Prof. Jenn Wortman Vaughan

April 18, 2012

Lecture 6

Reminders & Announcements

- Homework 2 will be posted on the course website by tomorrow and will be due on Monday, April 30
- You should be able to start the first half of the problems after this class and the rest after Monday's class

Today

- Expectation of a function of a random variables
- Variance

- Joint probability mass functions
- Conditional probability mass functions

Functions of Random Variables

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- How can we calculate $p_Y(l)$ for some value l?

- Suppose X is equally likely to take on any value in the set $\{-2, -1, 0, 1, 2\}$ and $Y = |X|$. What are p_X and p_Y?

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- Let X be a random variable and suppose $Y = aX + b$ for scalars a and b. What is E[Y]?

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The standard deviation of X is denoted σ_X, and is the square root of the variance. σ_X has the same units as X.

Computing Variance

With probability 0.6 there is low traffic and it takes me 30 minutes to get to work. The rest of the time there is high traffic and it takes me 50 minutes. Let X be the number of minutes of my commute.

What is E[X]? What is var(X)? What is σ_X?

Computing Variance

Let X be a random variable, and let $Y = aX + b$ for some scalar values a and b.

We know that $E[Y] = aE[X] + b$.

What is var(Y)? What about σ_Y?

Variance of the Bernoulli

Consider a Bernoulli random variable X with parameter p.

We know that the probability mass function is defined as:

$$p_X(1) = p$$

$$p_X(0) = 1 - p$$

What is var(X)? What is σ_X?

Multiple Random Variables

Joint PMFs

- Let X and Y be discrete random variables associated with the same experiment. The joint PMF of X and Y is denoted $p_{X,Y}$. For any values k of X and l of Y,

$$p_{X,Y}(k, l) = P(\{X=k\} \cap \{Y=l\}) = P(X=k, Y=l)$$

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    X = person's height, Y = person's weight.

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- Examples:
  - Experiment: Choose a random person in the class.
    X = person's height, Y = person's weight.
  - Experiment: Run a randomized algorithm.
    X = execution time, Y = amount of memory used.

Tabular Representation

X	Y	l_1	l_2	l_3	l_4
	k_1	0.1	0.1	0	0.2
X	k_2	0.05	0.05	0.1	0
	k_3	0	0.1	0.2	0.1

Tabular Representation

		Y			
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Is this a valid joint PMF? How do we know?

Marginal PMFs

We can compute the PMFs of X and Y from their joint PMF:

	l_1	l_2	l_3	l_4
k_1	0.1	0.1	0	0.2
k_2	0.05	0.05	0.1	0
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Marginal PMFs

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$$p_X(k) = \sum_l p_{X,Y}(k,l)$$

	l_1	l_2	l_3	l_4	$p_X(k)$
k_1	0.1	0.1	0	0.2	0.4
k_2	0.05	0.05	0.1	0	0.2
k_3	0	0.1	0.2	0.1	0.4

Marginal PMFs

We can compute the PMFs of X and Y from their joint PMF:

$$p_X(k) = \sum_l p_{X,Y}(k,l) \qquad p_Y(l) = \sum_k p_{X,Y}(k,l)$$

	l_1	l_2	l_3	l_4	$p_X(k)$
k_1	0.1	0.1	0	0.2	0.4
k_2	0.05	0.05	0.1	0	0.2
k_3	0	0.1	0.2	0.1	0.4
$p_Y(l)$	0.15	0.25	0.3	0.3	

Expectation

Expectation of a function: Let X and Y be random variables, and *f* be an arbitrary function of X and Y. Then

$$E[f(X,Y)] = \sum_{k} \sum_{l} f(k,l)\, p_{X,Y}(k,l)$$

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Linearity of expectation: Let X and Y be random variables. Then for any scalars a, b, and c,

$$\mathrm{E}[a\mathrm{X} + b\mathrm{Y} + c] = ???$$

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Linearity of expectation: Let X and Y be random variables. Then for any scalars *a*, *b*, and *c*,

$$\mathrm{E}[a\mathrm{X} + b\mathrm{Y} + c] = a\mathrm{E}[\mathrm{X}] + b\mathrm{E}[\mathrm{Y}] + c.$$

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All of this extends to more than two random variables too.

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- Joint PMFs:

$$
\begin{aligned}
p_{X,Y,Z}(k, l, m) &= P(X=k, Y=l, Z=m) \\
&= P(\{X=k\} \cap \{Y=l\} \cap \{Z=m\})
\end{aligned}
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- Joint PMFs:

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- Expectation of a function:

$$E[f(X,Y,Z)] = \sum_k \sum_l \sum_m f(k,l,m)\, p_{X,Y,Z}(k,l,m)$$

etc.

Three or More Random Variables

An office decides to do a gift exchange for their annual holiday party. Everyone in the office purchases a gift and places it in a box. Each person is then given a random gift from the box. What is the expected number of people who get back the gift that they purchased?

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Note that we used the fact that the expectation of a sum of random variables is equal to the sum of their expectations.

Independence is not required for this!

Three or More Random Variables

- What is the expected value of a binomial random variable with parameters p and n?

Conditioning

- The conditional PMF of X given Y is denoted $p_{X|Y}$. For any values k of X and l of Y,

$$p_{X|Y}(k \mid l) = P(\{X=k\} \mid \{Y=l\}) = P(X=k \mid Y=l)$$

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- We can compute this PMF using the definition of conditional probability

Conditioning

- We can compute conditional probabilities by normalizing the values in a particular row or column

	l_1	l_2	l_3	l_4
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$$p_{X|Y}(k \mid l_2) = \frac{p_{X,Y}(k, l_2)}{p_Y(l_2)}$$

Conditioning

Conditional PMFs frequently give us a convenient method of calculating joint PMFs. Using the multiplication rule,

$$p_{X,Y}(k,l) = p_X(k)p_{Y|X}(l|k) = p_Y(l)p_{X|Y}(k|l)$$

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(We'll continue with some examples next time…)