CS 112: Modeling Uncertainty in Information Systems

Prof. Jenn Wortman Vaughan
June 4, 2012
Lecture 17
Reminders & Announcements

• Homework 5 is due in section this Friday, June 8

• Course evaluations are available for you to complete online – we’ll end a little early on Thursday in case you want to use that time to complete them

• Friday’s section will be a review session for the final, so bring any questions that you have
Final Exam

• Tuesday, June 12, 11:30am – 2:30pm in this room

• One double-sided sheet of hand-written notes allowed

• No other notes, books, calculators, cell phones, etc.

• Best way to study is to practice solving problems like the end-of-chapter problems in the book (solutions available on the book’s website)
You should be able to…

- Figure out when and how to apply the basic rules and definitions of probability, including
  - the multiplication rule
  - Bayes’ rule
  - the total probability theorem
  - the total expectation theorem
  - the definition of conditional probability
  - the definition of independence
  - the counting principle
  - the definitions of expectation and variance
You should be able to...

• Translate word problems into math

• Define the appropriate random variables or events

• Be able to state the quantity that you must solve for in terms of these random variables or events

• Recognize common random variables (Bernoulli, binomial, and geometric) in problem descriptions
Topics Covered

- Sample spaces, events, and basic probability (Chapter 1)
- Discrete and continuous random variables (Chapters 2–3)
- Covariance and correlation (4.2)
- Markov, Chebyshev, law of large numbers (5.1, 5.2)
- Information, entropy, coding (as covered in class)
- MAP and maximum likelihood (as covered in class)
- Naive Bayes (external reading linked from website)
- Markov chains (7.1–7.4; see assignments on website)

You are responsible for any material that was covered in the reading, in class (including this week), or in section!
Back to Markov Chains...
Steady-State Convergence Theorem

**Theorem:** Consider any Markov chain with a single recurrent class, which is not periodic. There are unique values $\pi_1, \ldots, \pi_m$ that satisfy the balance equations:

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{k,j} \quad \text{for } j = 1, \ldots, m$$

$$\sum_{k=1}^{m} \pi_k = 1$$

and for each $j$, $\pi_j$ is the long term fraction of time that the state is $j$. These are called the steady-state probabilities.
Convergence of Markov Chains

• What if there is only one recurrent class but it is periodic?
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  • Limit of $P(X_n = i)$ is not well defined, but balance equations still give us the **long term frequencies**
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• What if there are multiple recurrent states?
  • Once a recurrent state is entered, the same ideas apply
  • How likely are we to enter each recurrent state?
Absorption
The Gambler’s Ruin

A gambler plays rounds of games in a casino. At each round, he wins $1 with probability $p$ and loses $1$ with probability $1-p$, independent of what happens in other rounds. He continues playing until he either accumulates a target amount of money $m$, or loses all of his money.

- How can we represent this scenario as a Markov chain?
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- What are the recurrent classes?
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• How can we represent this scenario as a Markov chain?
• What are the recurrent classes?
• What is the probability that the gambler accumulates $m$?
Absorbing States

• An absorbing state in a Markov chain is a single-state recurrent class — that is, any state \( i \) such that \( p_{i,i} = 1 \)
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What do we know about the values $a_i$?
Absorption Probability Equations

**Theorem:** Consider a Markov chain in which each state is either absorbing or transient. Fix a particular absorbing state $s$. Then the probabilities $a_i$ of eventually reaching state $s$ after starting at state $i$ are the unique solutions to the following system of equations:

\[
\begin{align*}
    a_s &= 1 \\
    a_i &= 0 & \text{for all absorbing } i \neq s \\
    a_i &= \sum_{j=1}^{m} p_{i,j} a_j & \text{for all transient } i
\end{align*}
\]
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(we’ll just do the case where $p = .5$, but you can read the general solution in example 7.11 in the text)
Gambler’s Ruin

\[ m = 1000 \quad p = 0.4999 \]
Gambler’s Ruin

\[ m = 1000 \quad \text{and} \quad p = 0.499 \]
Gambler’s Ruin

\[ m = 1000 \quad p = 0.49 \]
Gambler’s Ruin

\[ m = 1000 \quad p = 0.4 \]