CS 112: Modeling Uncertainty in Information Systems

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Lecture 15

Reminders & Announcements

- Homework 4 is due one week from today
- Midterm regrade requests must be submitted by ~~Monday~~ Tuesday (Monday is a holiday!); see Piazza for information on the grading of problem 1

Markov Chains

Reasoning About Complex Scenarios

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  - The sequence of daily prices of a stock
  - The sequence of websites a web surfer visits (PageRank)
  - The number of packets in a network buffer over time
  - The sequence of words in an “English-looking” document

Markov Chains

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A Markov chain is specified by:

- A set of possible states $S = \{1, 2, \ldots, m\}$, such that at each time t, the current state $X_t \in S$
- A distribution over the initial state X_0
- A set of transition probabilities $p_{i,j}$ where

$$p_{i,j} = P(X_{t+1} = j \mid X_t = i)$$

A Simple Example

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

- What are the states?
- What are the transition probabilities?

Representing Transition Probabilities

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- The best representation depends on the problem you are trying to solve

The Markov Property

The key independence assumption we make is called the Markov property: For all times t, X_{t+1} is conditionally independent of $X_1, X_2, \ldots, X_{t-1}$ given X_t.

The Markov Property

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

Suppose that if the router remains offline for four straight days, I get it (temporarily) repaired, resetting it to the online state.

Can we still represent this experiment as a Markov chain?

n-Step Transitions

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- We can efficiently compute the n-step transition probability, $P(X_n = j \mid X_0 = i)$, using the Total Probability Theorem, giving us the recursive formula

$$P(X_n = j \mid X_0 = i) = \sum_{k=1}^{m} p_{k,j} P(X_{n-1} = k \mid X_0 = i)$$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 1) = 1$ $\qquad$ $P(X_0 = 2 \mid X_0 = 1) = 0$

n-Step Transitions

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$P(X_0 = 1 \mid X_0 = 1) = 1$ $\qquad$ $P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 1 \mid X_0 = 1) = 0.8$ $\qquad$ $P(X_1 = 2 \mid X_0 = 1) = 0.2$

n-Step Transitions

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$P(X_0 = 1 \mid X_0 = 1) = 1$

$P(X_1 = 1 \mid X_0 = 1) = 0.8$

$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

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$P(X_0 = 1 \mid X_0 = 1) = 1$

$P(X_1 = 1 \mid X_0 = 1) = 0.8$

$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_3 = 1 \mid X_0 = 1) = 0.752$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

$P(X_3 = 2 \mid X_0 = 1) = 0.248$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

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$P(X_1 = 1 \mid X_0 = 1) = 0.8$

$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_3 = 1 \mid X_0 = 1) = 0.752$

$P(X_4 = 1 \mid X_0 = 1) = 0.7504$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

$P(X_3 = 2 \mid X_0 = 1) = 0.248$

$P(X_4 = 2 \mid X_0 = 1) = 0.2496$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 1) = 1$

$P(X_1 = 1 \mid X_0 = 1) = 0.8$

$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_3 = 1 \mid X_0 = 1) = 0.752$

$P(X_4 = 1 \mid X_0 = 1) = 0.7504$

$P(X_5 = 1 \mid X_0 = 1) = 0.7501$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

$P(X_3 = 2 \mid X_0 = 1) = 0.248$

$P(X_4 = 2 \mid X_0 = 1) = 0.2496$

$P(X_5 = 2 \mid X_0 = 1) = 0.2499$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$ $P(X_0 = 2 \mid X_0 = 2) = 1$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_3 = 1 \mid X_0 = 2) = 0.744$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

$P(X_3 = 2 \mid X_0 = 2) = 0.256$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_3 = 1 \mid X_0 = 2) = 0.744$

$P(X_4 = 1 \mid X_0 = 2) = 0.7488$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

$P(X_3 = 2 \mid X_0 = 2) = 0.256$

$P(X_4 = 2 \mid X_0 = 2) = 0.2512$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_3 = 1 \mid X_0 = 2) = 0.744$

$P(X_4 = 1 \mid X_0 = 2) = 0.7488$

$P(X_5 = 1 \mid X_0 = 2) = 0.7498$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

$P(X_3 = 2 \mid X_0 = 2) = 0.256$

$P(X_4 = 2 \mid X_0 = 2) = 0.2512$

$P(X_5 = 2 \mid X_0 = 2) = 0.2502$

Properties of States

Accessibility

- State j is said to be accessible from state i if for some $n \geq 0$, the n-step transition probability from i to j is positive.

Recurrence vs. Transience

- State i is said to be recurrent if for every state j that is accessible from i, i is also accessible from j.
- If i is not recurrent, then it is said to be transient.

Recurrence vs. Transience

- State i is said to be recurrent if for every state j that is accessible from i, i is also accessible from j.
- If i is not recurrent, then it is said to be transient.
- The set of all states accessible from a recurrent state form a recurrent class. Note that all of the states in a recurrent class are accessible from each other.

Recurrence vs. Transience

- If a Markov chain starts in a recurrent state, the state will stay within that recurrent class forever.

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- If a (finite state) Markov chain starts in a transient state, the state will pass through zero or more additional transient states before ending up in a recurrent class.

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- If a (finite state) Markov chain starts in a transient state, the state will pass through zero or more additional transient states before ending up in a recurrent class.

- These ideas will allow us to answer questions about the long term behavior of Markov chains in the next class.