CS 112: Modeling Uncertainty in Information Systems

Prof. Jenn Wortman Vaughan

May 9, 2012

Lecture 11

Reminders & Announcements

- Homework 3 was posted on Monday and is due on Friday, May 18
- Prof. Vaughan will be traveling tomorrow and will not hold office hours this week

Today…

- Joint probability density functions
- Conditional probability density functions

We'll go over many of the similarities between joint & conditional PDFs and joint & conditional PMFs, but will leave others for the reading.

See Chapter 3 for full details!

Joint Probability Density Functions

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- For any a, any $b \geq a$, any c, and any $d \geq c$,

$$P(a \leq X \leq b \text{ and } c \leq Y \leq d) = \int_c^d \int_a^b f_{X,Y}(k,l)\,dk\,dl$$

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- Continuous case:

$$P(a \le X \le b \mid A) = P(\{a \le X \le b\} \mid A) = \int_a^b f_{X|A}(k)\,dk$$

Conditioning on a Random Variable

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- This can be used to derive an analog to Bayes' rule too

Joint CDFs

- Same definition for discrete and continuous:

$$F_{X,Y}(k,l) = P(X \le k, Y \le l)$$

though how we calculate this will differ…

Example: Circular Uniform PDF

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- What is $f_Y(l)$?

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- What is $f_{X,Y}(k, l)$?
- What is $f_Y(l)$?
- What is $f_{X|Y}(k|l)$?

Independence

- Discrete case: X and Y are independent iff for all *k*, *l*

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- Analogs of the other independence tests can be used in the continuous case too

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Are X and Y independent?

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Are X and Y independent?

What if the target was a square?

Conditional Expectation

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For an event A, E[X | A] is defined similarly...

Total Expectation Theorem

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$$E[X] = \sum_{i=1}^{n} P(A_i) E[X \mid A_i]$$

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When Y is continuous:

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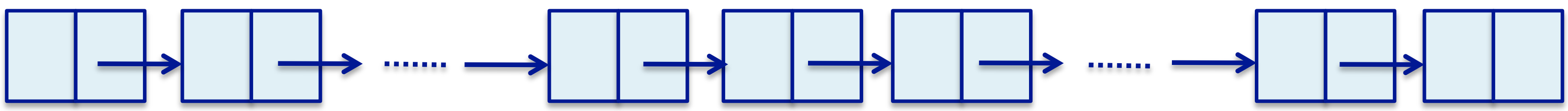
These hold regardless of whether X is discrete or continuous

Searching a Sorted Linked List

Consider a (very long) singly linked list with entries sorted in ascending order.



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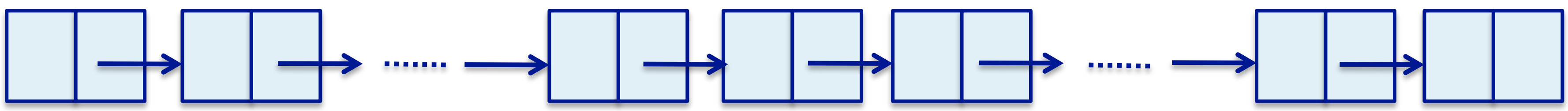

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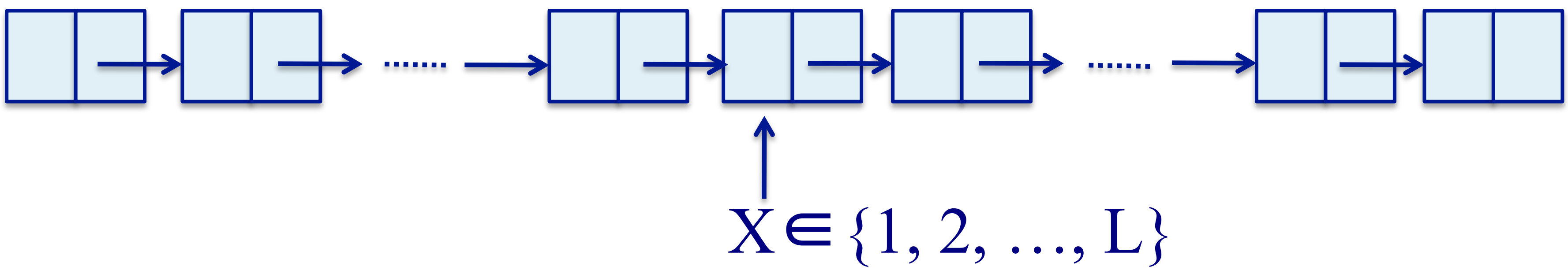

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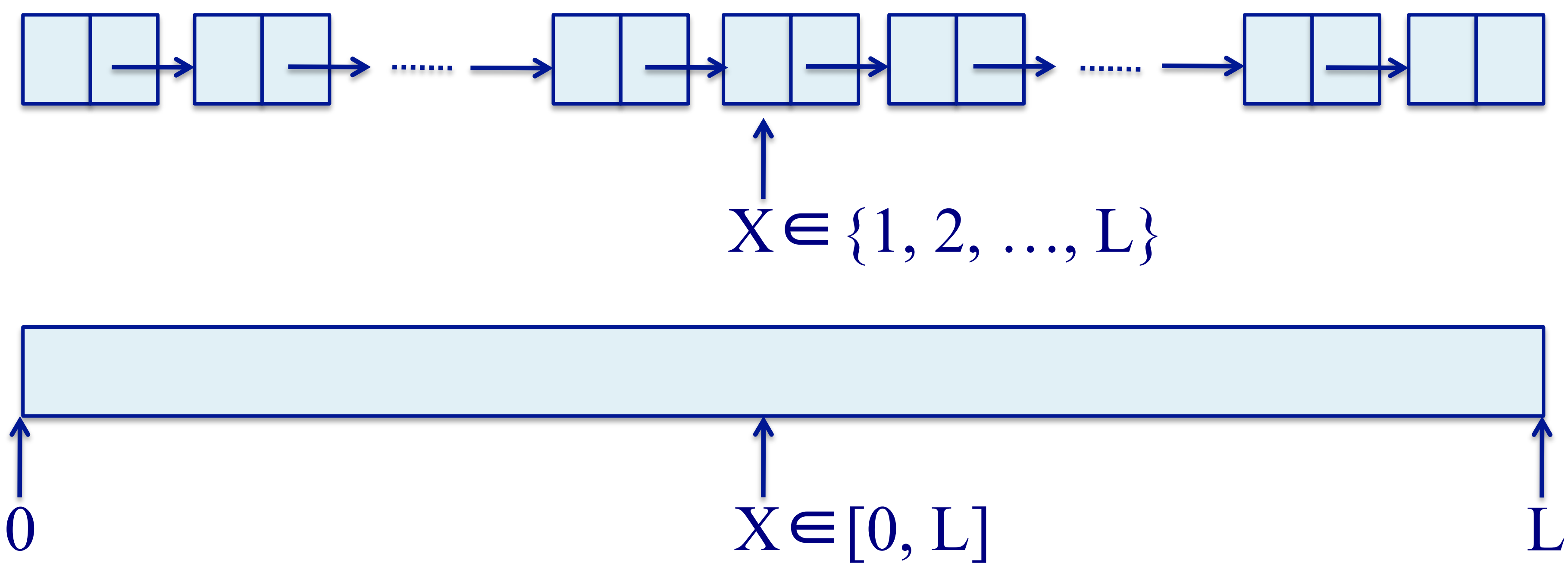
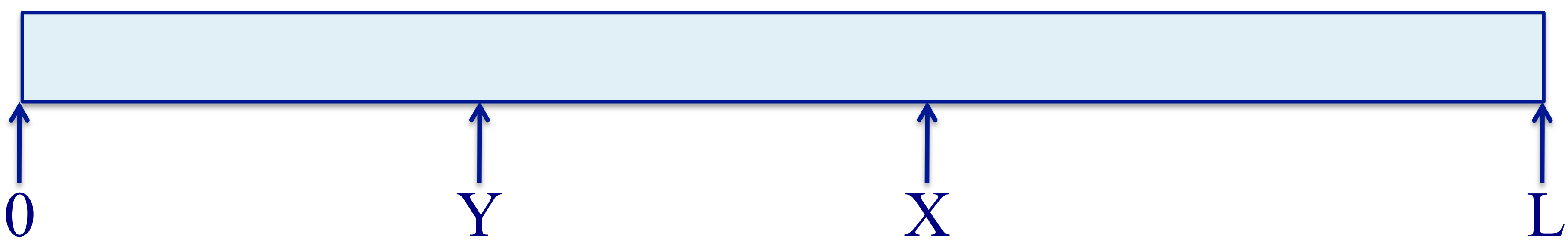


Searching a Sorted Linked List

Let X be a random variable denoting the location of the last item we found. Let Y denote the location of the next item that we need to search for. We can either search for the next item starting at position 0 or starting at position X.



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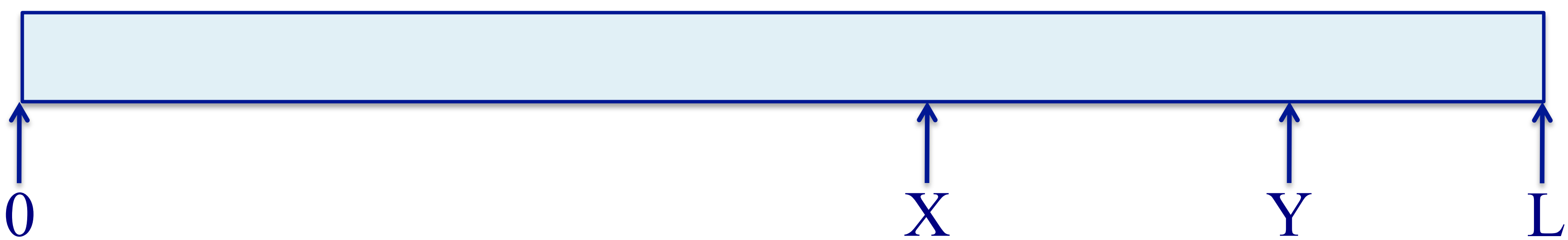

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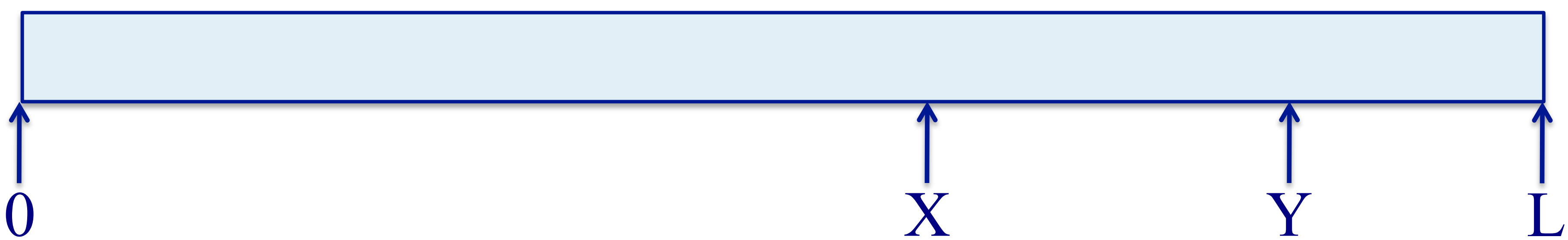

If X and Y are independent and uniform on [0, L], then what is the expected length of our search?

Searching a Sorted Linked List

- We could have solved this problem using discrete random variables, but using continuous random variables makes the math a little nicer and gets a good approximation

- You'll see another example of this idea on homework 3

Other Analogs

- There are continuous analogs of other ideas we have discussed too (the total probability theorem, linearity of expectation, etc.)

- Read Chapter 3 to familiarize yourself with these!