# CS112: Modeling Uncertainty in Information Systems
# Homework 5

Due Friday, June 8, at the beginning of section

Please refer to the course academic integrity policy for collaboration rules. In particular, be sure to include a list of anyone with whom you have discussed the assignment. Remember that you will be graded on both the correctness and the clarity of your solutions. You will not receive credit for a solution if the grader can't read your writing or understand your argument. Only a subset of the problems will be graded for credit. Show all of your work. Start early!

IMPORTANT: For each problem, be sure to clearly define any events or random variables that you use in your solution!!

1. A stream of data is sent over a noisy channel, where the probability that a bit is accidentally flipped is 0.001. You receive a block of 1000 bits. Let $X$ be a random variable representing the number of bits that have been flipped.
   (a) Use the Markov inequality to find an upper bound on the probability that the block has 5 or more flipped bits.
   (b) Use the Chebyshev inequality to bound the same probability.

2. Suppose a person goes for a heart check-up and the doctor tries to ascertain whether that person has a heart condition based on three characteristics: the patient's gender, blood pressure, and electrocardiograph (ECG) reading.

   Let $M$ be the event that the patient is male and $F$ be the event that the patient is female.

   Let $H$ be the event that the patient has high blood pressure and $L$ be the event that the patient has low blood pressure (i.e., does not have high blood pressure).

   Let $N$ be the event that the ECG reading is normal and $A$ be the event that it is abnormal.

   Finally, let $C$ be the event that the patient has a heart condition.

   We are given the following information:

$$\begin{array}{llll} P(C) = 0.1 & P(M|C^c) = 0.8 & P(H|C^c) = 0.6 & P(A|C^c) = 0.5 \\ & P(M|C) = 0.6 & P(H|C) = 0.7 & P(A|C) = 0.8 \end{array}$$

   Assume the patient's gender, blood pressure, and ECG reading are all conditionally independent of each other given the presence or absence of a heart condition.

   (a) A female patient has low blood pressure and an abnormal ECG. What are the maximum likelihood and MAP hypotheses regarding whether or not she has a heart condition?
   (b) What is the posterior probability of this patient having a heart condition?
   (c) A male patient has low blood pressure and a normal ECG. What are the maximum likelihood and MAP hypotheses regarding whether or not he has a heart condition?

3. Consider a Markov chain with the following transition probability ($P$) matrix for states {1,2,3,4,5,6}:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0 & 0.3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0.1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Draw the state transition diagram of the Markov chain.

(b) For each of the six states, state whether it is recurrent or transient.

(c) What are the recurrent classes of this Markov chain?

(d) For each recurrent class, state whether it is periodic or aperiodic. If periodic, what is the period?

(e) Let $X_t$ denote the state of the Markov chain at time $t$. What is $P(X_3 = 5|X_0 = 3)$?

(f) In the long run, what proportion of the time will the Markov chain be in each state?

4. A workstation tries to transmit frames through the Ethernet. Suppose that whether or not collision occurs in the current transmission depends on the result of the last two transmissions the workstation had. In particular, if collisions occurred in both of the past two transmissions, then with probability 0.8 a collision will occur in the current transmission; if a collision occurred in the last transmission but not the transmission before the last one, then a collision will occur in the current transmission with probability 0.5; if no collision occurred in the last transmission, then a collision will occur in the current transmission with probability 0.2.

   (a) Define appropriate states in order to model this scenario as a Markov Chain.

   (b) Draw the state transition diagram.

   (c) Find the transition probability matrix for the states defined in part (a).

   (d) What fraction of frames suffer a collision in the long run?

5. Consider two identical looking coins. One is fair so the probability of a head or tail resulting from a toss is equally likely. The second coin is biased, with the probability of a toss resulting in a head denoted by $p$, and the probability of a toss resulting in a tail $1-p$.

   A sequence of heads and tails is generated as follows. The "current coin" is tossed. If the result is a tail, we switch to the other coin for the next toss. If the result is a head, we toss the same coin to generate the next result. Define a discrete time Markov Chain where the state corresponds to the current coin, i.e. state 1 = biased, and state 2 = fair.

   (a) Draw the Markov Chain.

   (b) Write the transition probability matrix for this MC.

   (c) What is the solution for the stationary state probabilities?

   (d) Based on the result from part b, what is the long range fraction of coin tosses that result in heads?

   (e) Given that the result of the last toss was a head, what is the probability that the next toss result will be a head?