CS 112: Computer System Modeling Fundamentals

Prof. Jenn Wortman Vaughan

April 26, 2011

Lecture 9

Reminders & Announcements

- The course midterm is one week from today in class
  - The exam will cover all of Chapters 1–3 except for 3.3
  - Emphasis will be on Chapters 1 and 2
  - One double-sided sheet of hand-written notes allowed
  - No other notes, books, calculators, cell phones, etc.
  - Best way to prep is to practice problems from the book
- Homework 3 will be posted by Thursday and due in two weeks – also will be good practice for the exam!

Last Time

- Relationship between exponential random variables and geometric random variables

- Joint probability density functions

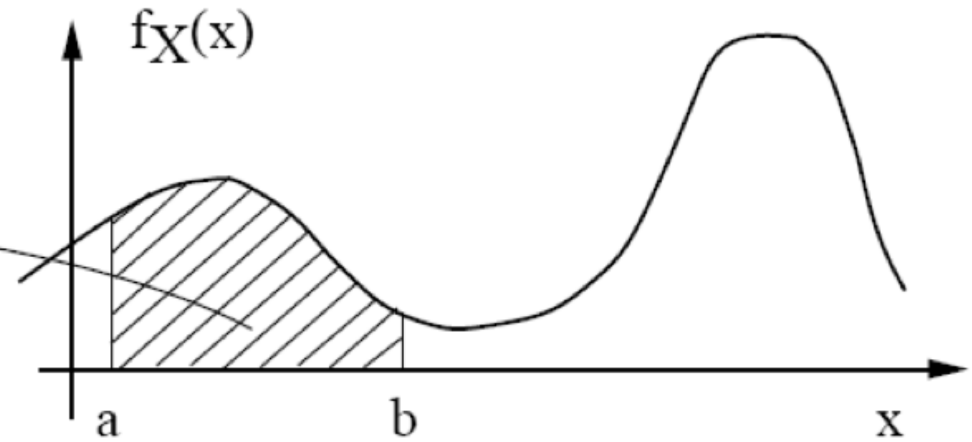
Today…

- More examples of how to work with continuous random variables and joint PDFs
- Independence, Bayes' rule, and conditional expectation for continuous random variables
- The Total Expectation Theorem for continuous random variables & an application to searching sorted linked lists

The Probability Density Function

The probability density function (or PDF) is denoted f_X.

$$P(a \le X \le b) = \int_a^b f_X(k)\,dk$$

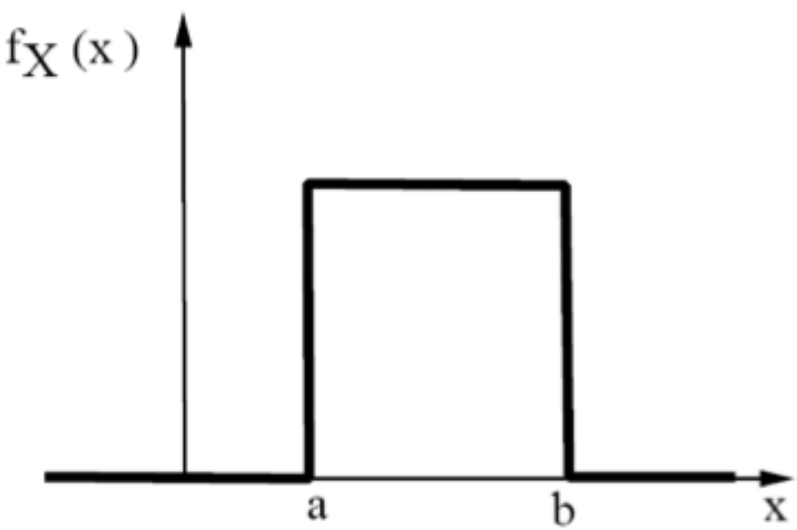


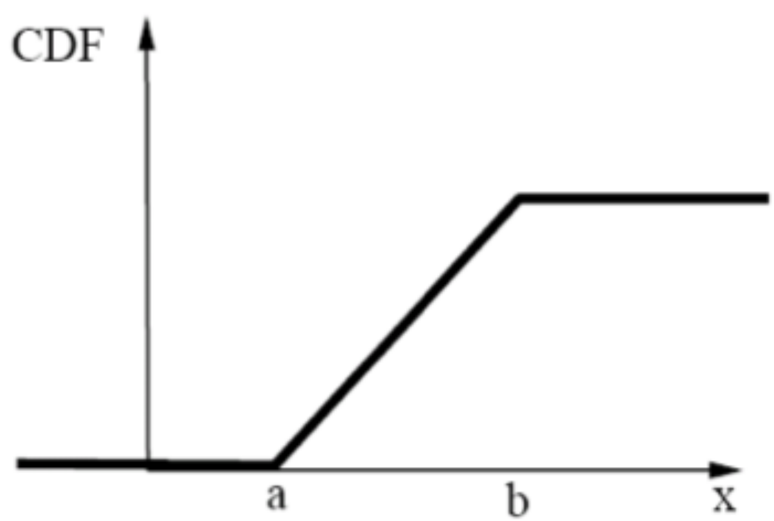
To satisfy normalization, we need $\int_{-\infty}^{\infty} f_X(k)\,dk = 1$

Cumulative Distribution Functions

A cumulative distribution function (CDF), denoted F_X, "accumulates" probability up to a certain value of X

$$F_X(k) = P(X \leq k)$$





Exponential Random Variables

Exponential random variables model the amount of time until an incident of interest takes place

- Length of time before a message arrives at the computer
- Length of time before a light bulb burns out

PDF: $f_X(k) = \lambda e^{-\lambda k}$

CDF: $F_X(k) = 1 - e^{-\lambda k}$

$E[X] = \lambda^{-1}$ $\quad$ $var(X) = \lambda^{-2}$

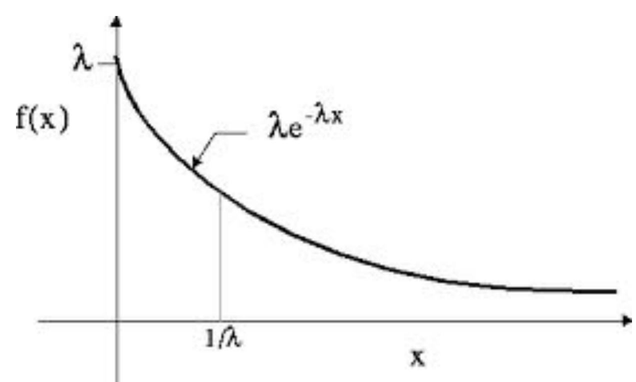


Figure 6. Exponential *pdf*

Joint PDFs

Joint density function: $f_{X,Y}(x,y)$

Marginalization: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

Conditional PDF: $f_{X|Y}(x \mid y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

Multiplication rule: $f_{X,Y}(x,y) = f_{X|Y}(x \mid y)\, f_Y(y)$

Example: Memoryless Variables

The time T until a light bulb burns out is an exponential random variable with parameter λ. Suppose you turn the light bulb on, leave the room, and come back t minutes later to find the light bulb still on. Let X be the additional time until the light bulb burns out. What is the CDF of X given that the light bulb was still on at time t?

Example: Memoryless Variables

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$$P(X \leq k \mid T > t) = 1 - e^{-\lambda k}$$

Same CDF as T!! Exponential RVs are memoryless.

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Geometric random variables satisfy this property too…

Independence

X and Y are independent if for all *x*, *y*

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$

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The analogs of the alternate tests hold too, e.g., X and Y are independent if for all *x*, and all *y* such that $f_Y(y) > 0$,

$$f_{X|Y}(x \mid y) = f_X(x)$$

Independence

Suppose you throw a dart at a circular target of radius r. Assume that you always hit the target, and you are equally likely to hit any point (x, y) on the target. Let X and Y denote the coordinates of the point that you hit.

Are X and Y independent?

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Are X and Y independent?

What if the target was a square?

Bayes' Rule

- Bayes' rule can also be extended in the natural way to hold for continuous random variables:

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- Alternate versions can be derived for the case when one of X and Y is discrete and the other continuous too…

Conditional Expectation

When X is discrete:

$$E[X \mid Y = y] = \sum_{x} x p_{X \mid Y}(x \mid y)$$

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For an event A, E[X | A] is defined similarly...

Total Expectation Theorem

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$$E[X] = \sum_{i=1}^{n} P(A_i) E[X \mid A_i]$$

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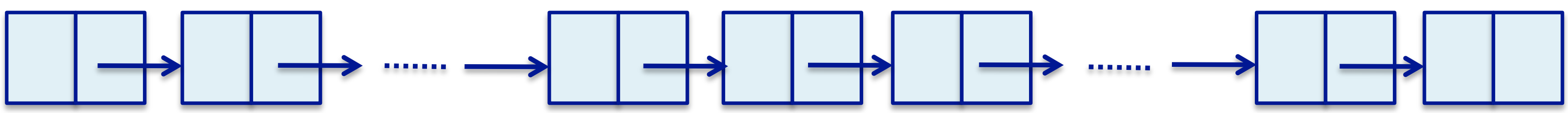
When Y is continuous:

$$E[X] = \int_{-\infty}^{\infty} f_Y(y) E[X \mid Y = y] \, dy$$

These hold regardless of whether X is discrete or continuous

Searching a Sorted Linked List

Consider a (very long) singly linked list with entries sorted in ascending order.



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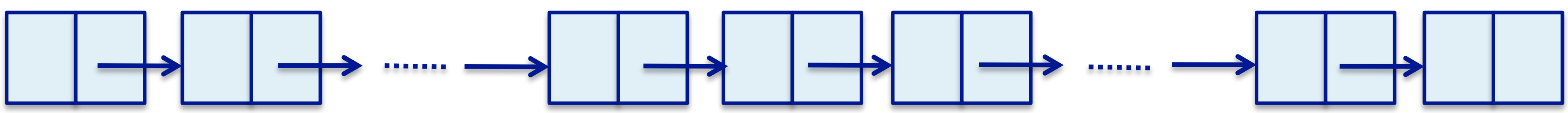

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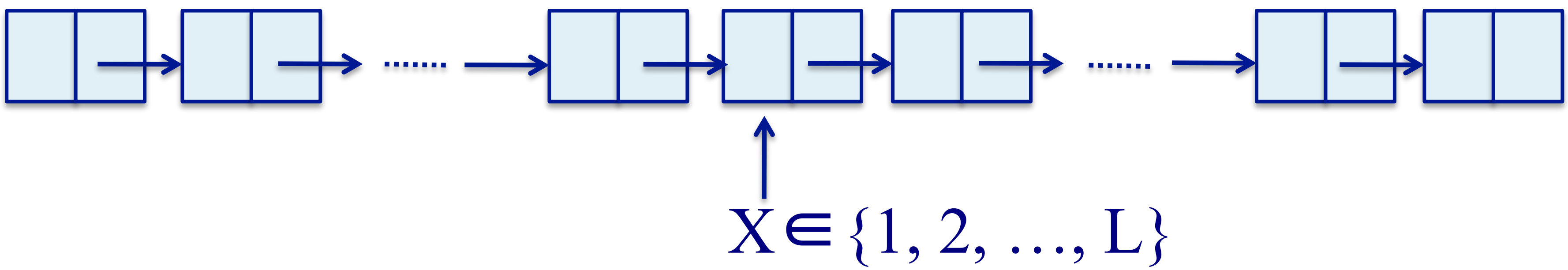

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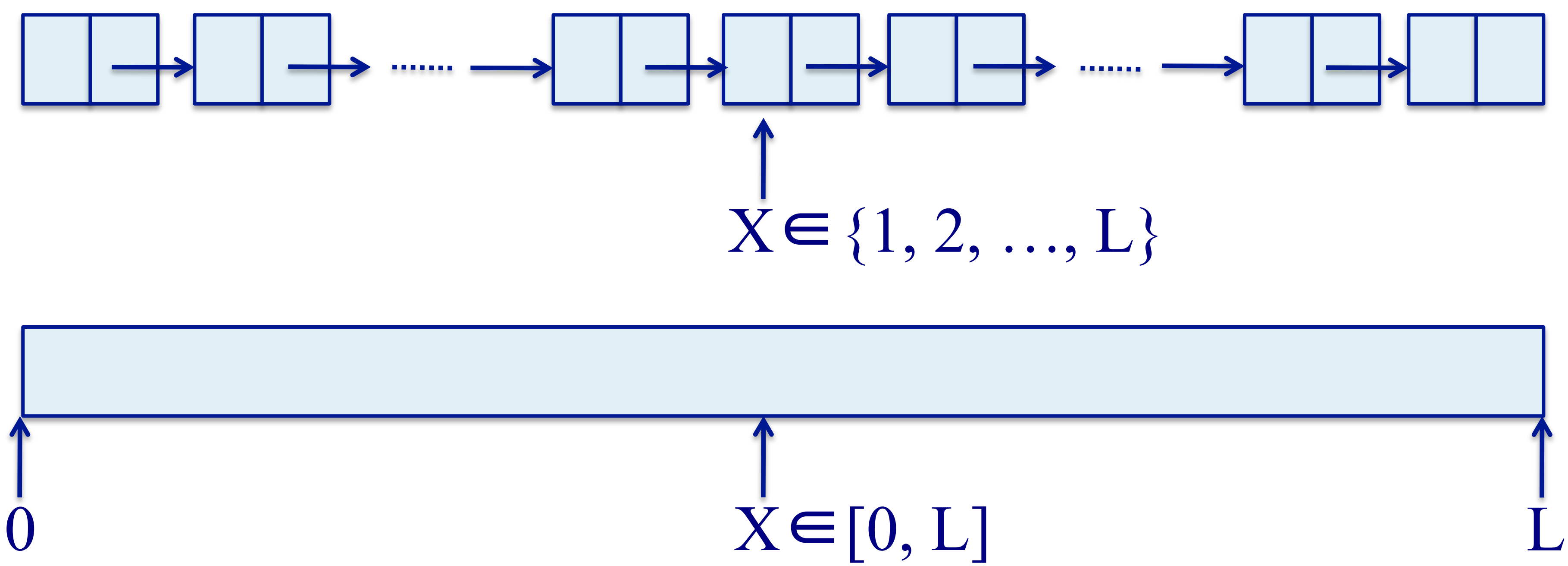
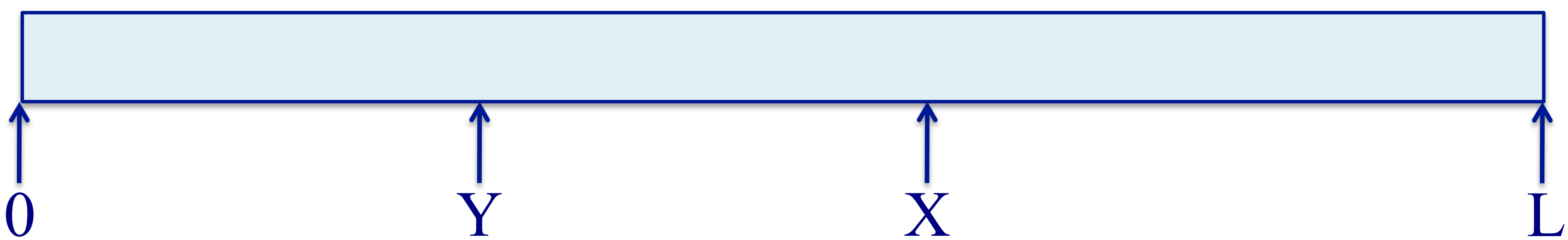


Searching a Sorted Linked List

Let X be a random variable denoting the location of the last item we found. Let Y denote the location of the next item that we need to search for. We can either search for the next item starting at position 0 or starting at position X.



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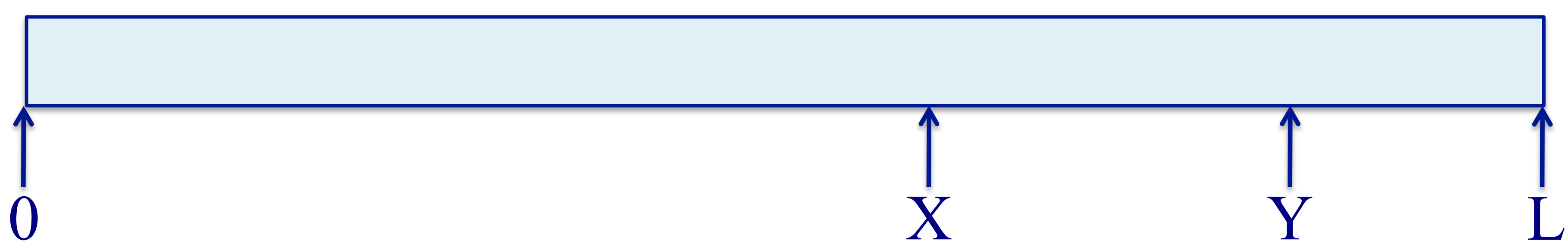

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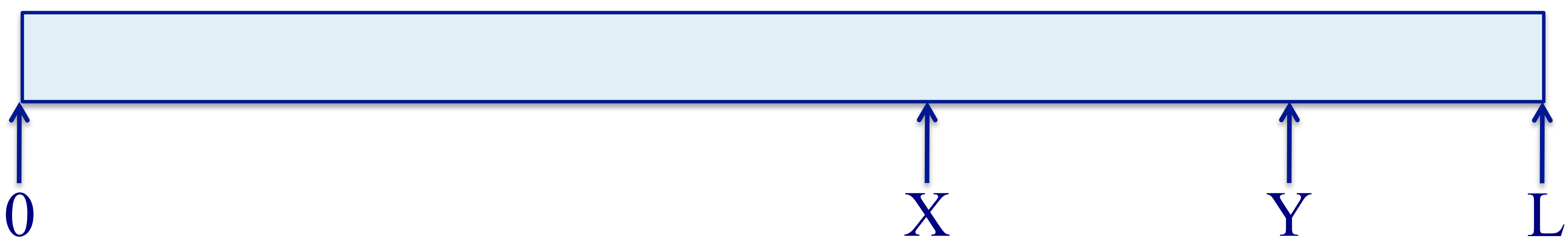

If X and Y are both uniform in [0, L], then what is the expected length of our search?

Total Expectation Theorem

For events $A_1, \ldots, A_n$ that partition the sample space:

$$E[X] = \sum_{i=1}^{n} P(A_i) E[X \mid A_i]$$

When Y is discrete:

$$E[X] = \sum_{y} p_Y(y) E[X \mid Y = y]$$

When Y is continuous:

$$E[X] = \int_{-\infty}^{\infty} f_Y(y) E[X \mid Y = y] \, dy$$

These hold regardless of whether X is discrete or continuous

(One important idea that we discussed here that was not on the slides is that any rules that apply to probabilities apply to conditional probabilities too since conditional probabilities obey the probability axioms… so for example we get the following from the first version of the total expectation theorem)

For events A_1, ..., A_n that partition the sample space and any event B:

$$E[X \mid B] = \sum_{i=1}^{n} P(A_i \mid B) E[X \mid A_i \cap B]$$