CS 112: Computer System Modeling Fundamentals

Prof. Jenn Wortman Vaughan

April 21, 2011

Lecture 8

Quiz #2

Reminders & Announcements

- Homework 2 is due in class on Tuesday – be sure to check the posted homework 1 solutions to see how you should structure your own homework solutions!!
- There is no discussion section tomorrow

Last Time

A bit more about joint PMFs

- Variance of sums of independent random variables
- Covariance and correlation

Continuous random variables

- Probability density functions (PDFs)
- Uniform continuous random variables
- Exponential random variables
- Cumulative distribution functions (CDFs)

Today

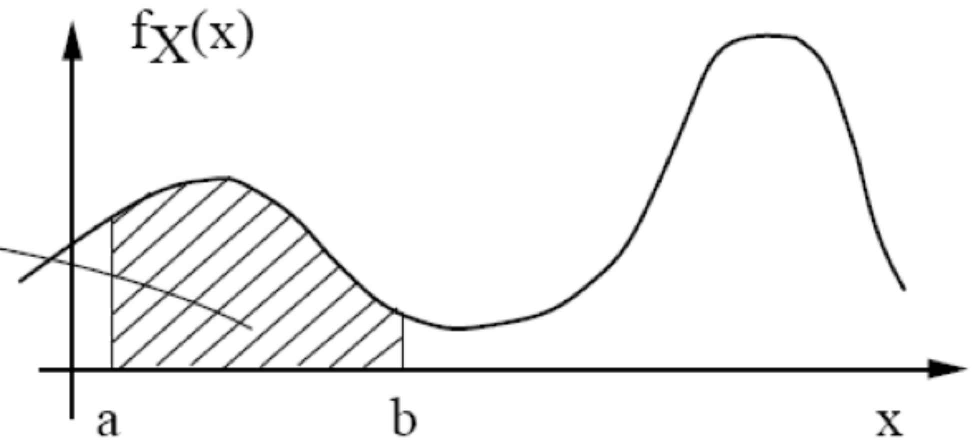
- Relationship between exponential random variables and geometric random variable

- Joint probability density functions

The Probability Density Function

The probability density function (or PDF) is denoted f_X.

$$P(a \le X \le b) = \int_a^b f_X(k)\,dk$$



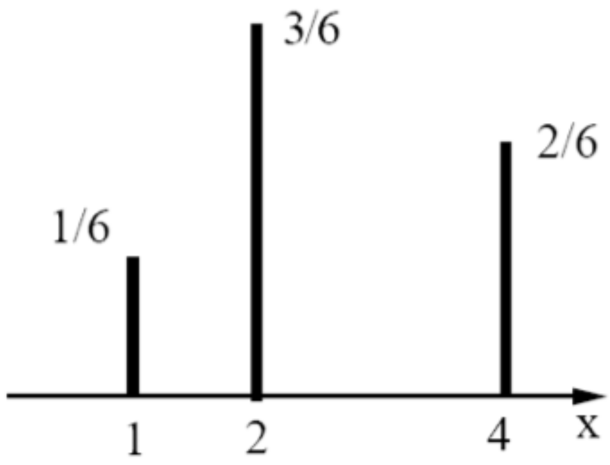
To satisfy normalization, we need $\int_{-\infty}^{\infty} f_X(k)\,dk = 1$

Cumulative Distribution Functions

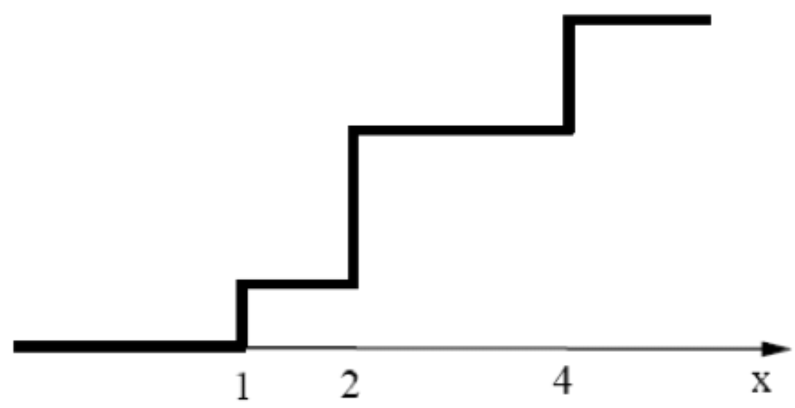
A cumulative distribution function (CDF), denoted F_X, "accumulates" probability up to a certain value of X

$$F_X(k) = P(X \leq k)$$

PMF:



CDF:



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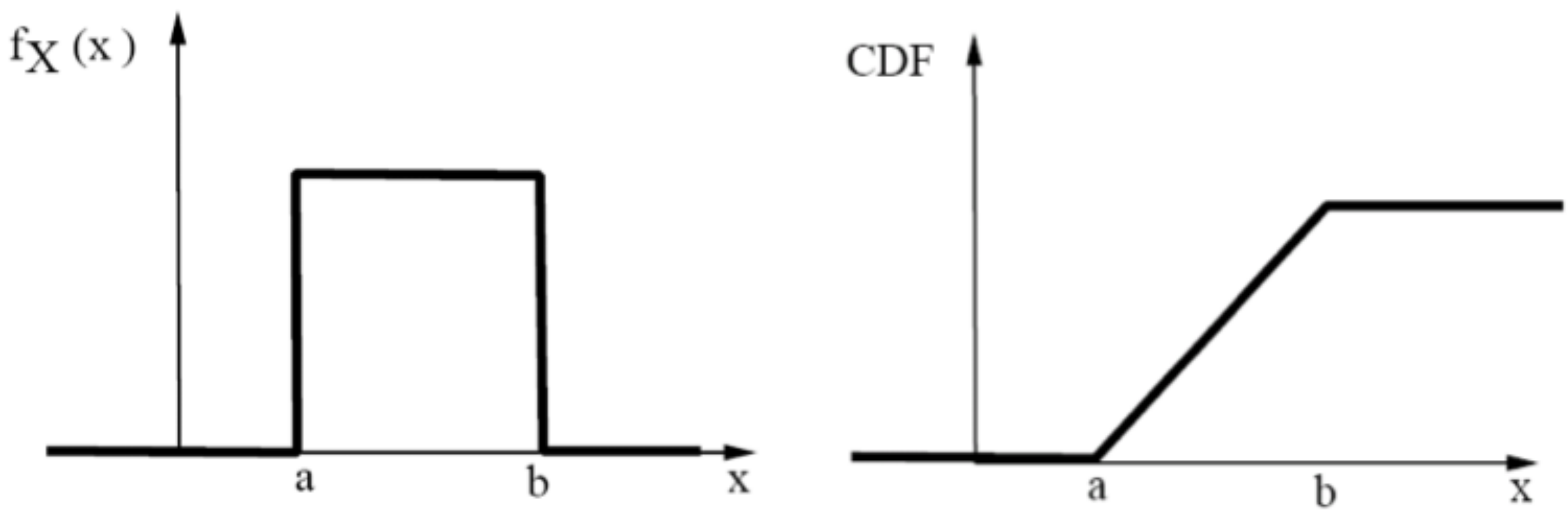

Exponential Random Variables

Exponential random variables model the amount of time until an incident of interest takes place

- Length of time before a message arrives at the computer
- Length of time before a light bulb burns out

PDF: $f_X(k) = \lambda e^{-\lambda k}$

$E[X] = \lambda^{-1}$ $\quad var(X) = \lambda^{-2}$

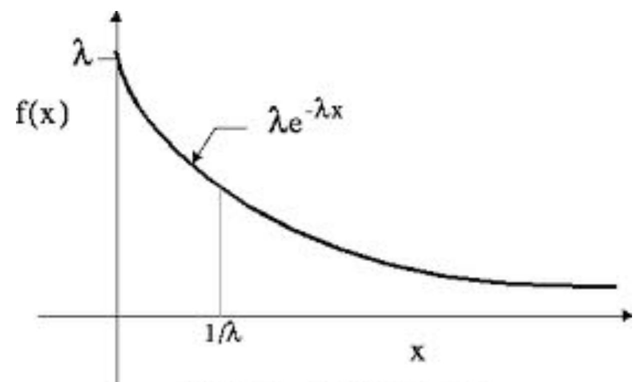

Figure 6. Exponential *pdf*

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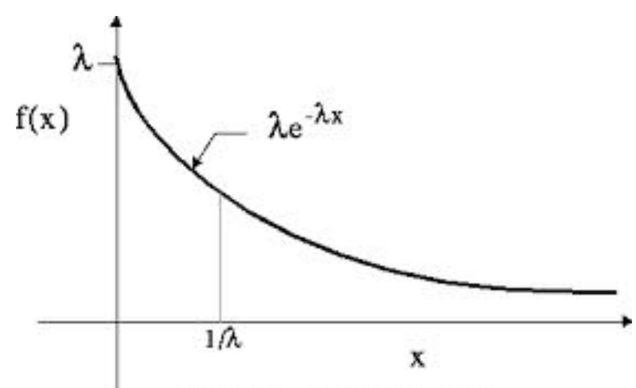

Figure 6. Exponential *pdf*

Can use the CDF to show a relationship between exponential random variables and geometric variables.

Exponential vs. Geometric

Let X be an exponential random variable with $f_X(t) = \lambda e^{-\lambda t}$.

$$F_X(t) = ???$$

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Suppose that we set $p = 1 - e^{-\lambda}$…

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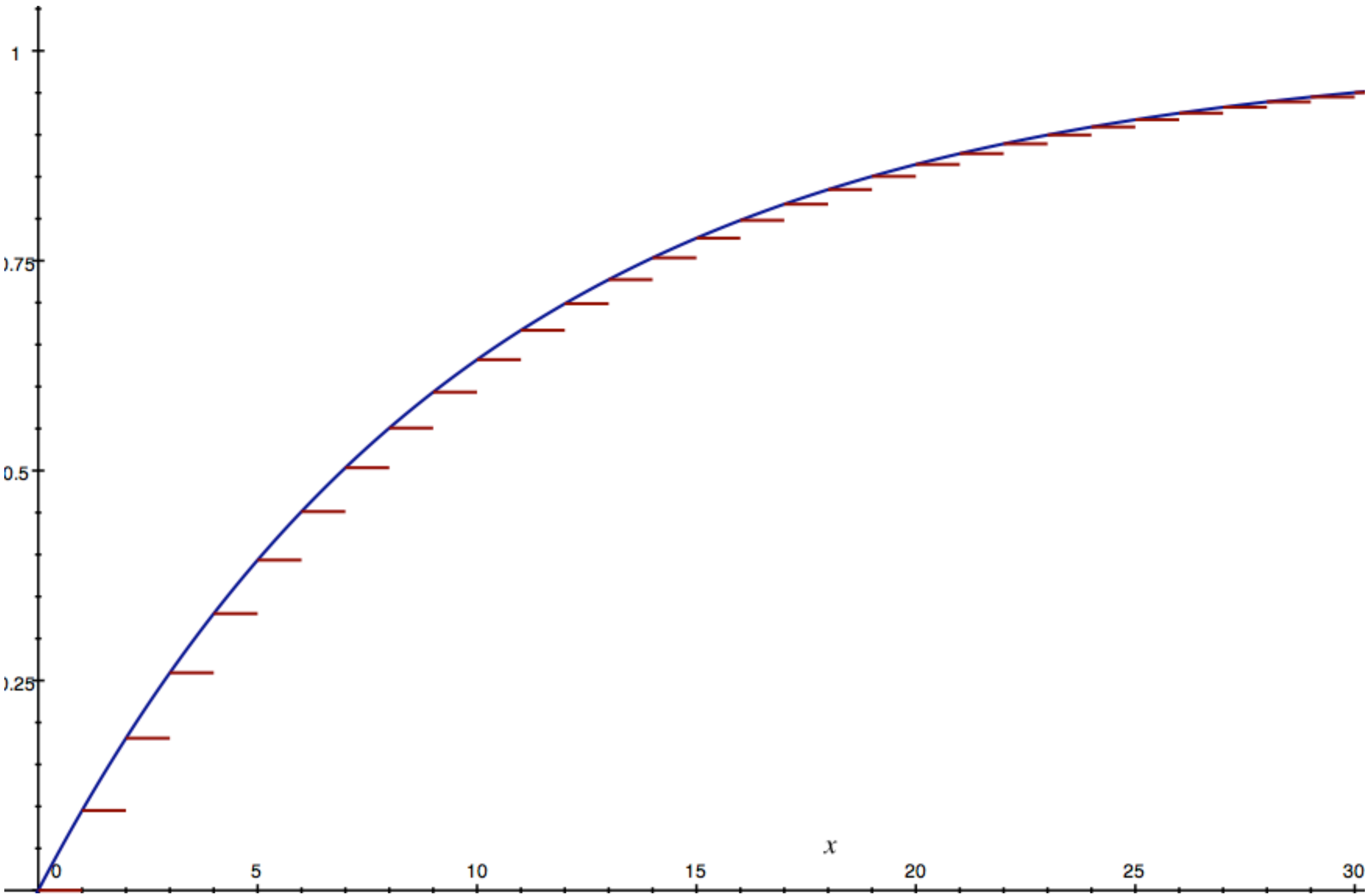

Joint PMFs vs. Joint PDFs

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Example: Circular Uniform PDF

Suppose you throw a dart at a circular target of radius r. Assume that you always hit the target, and you are equally likely to hit any point (x, y) on the target. Let X and Y denote the coordinates of the point that you hit.

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- What is $f_Y(y)$?
- What is $f_{X|Y}(x|y)$?

Example: Memoryless Variables

The time T until a light bulb burns out is an exponential random variable with parameter λ. Suppose you turn the light bulb on, leave the room, and come back t minutes later to find the light bulb still on. Let X be the additional time until the light bulb burns out. What is the CDF of X given that the light bulb was still on at time t?

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(we stopped midway through our discussion of memorylessness, but will pick up here next time)