CS 112: Computer System Modeling Fundamentals

Prof. Jenn Wortman Vaughan

April 19, 2011

Lecture 7

Reminders & Announcements

- Check your grade for homework 1 on Gradebook
- Homework 2 is due on Tuesday, April 26
- Check the homework 1 solutions to see how you should structure your own homework solutions – we'll be less generous about sloppy or confusing answers next time!!
- There is no Friday section this week

Last Time

What if we have multiple random variables associated with the same event?

- Joint PMFs
- Conditioning
- Independence

Remember, most of this was just new notation for concepts we already knew. When in doubt, think about the underlying events!

Today

A bit more about joint PMFs

- Variance of sums of independent random variables
- Covariance and correlation

Continuous random variables

- Probability density functions (PDFs)
- Uniform continuous random variables
- Exponential random variables
- Cumulative distribution functions (CDFs)

Independence

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$$p_Y(l) = p_{Y|X}(l \mid k)$$

To prove or disprove independence, it is enough to prove or disprove any one of these conditions.

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$$\mathrm{var}(X_1 + X_2 + \ldots + X_n) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \ldots + \mathrm{var}(X_n)$$

The Sample Mean

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How accurate is our estimate as a function of n?

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How accurate is our estimate as a function of n?

These ideas will come up again later in the course…

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- Let X be his height and Y be his weight. These have positive covariance or are positively correlated.
- Let X be the number of courses he is taking and Y be the average number of hours he sleeps each night. These have negative covariance or are negatively correlated.
- Let X be his height and Y be the last digit of his UID. These have (roughly) zero covariance or are uncorrelated.

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- What is cov(X, X)?
- Suppose X and Y are independent. What is cov(X, Y)?
- Suppose cov(X, Y) = 0. Are X and Y independent?

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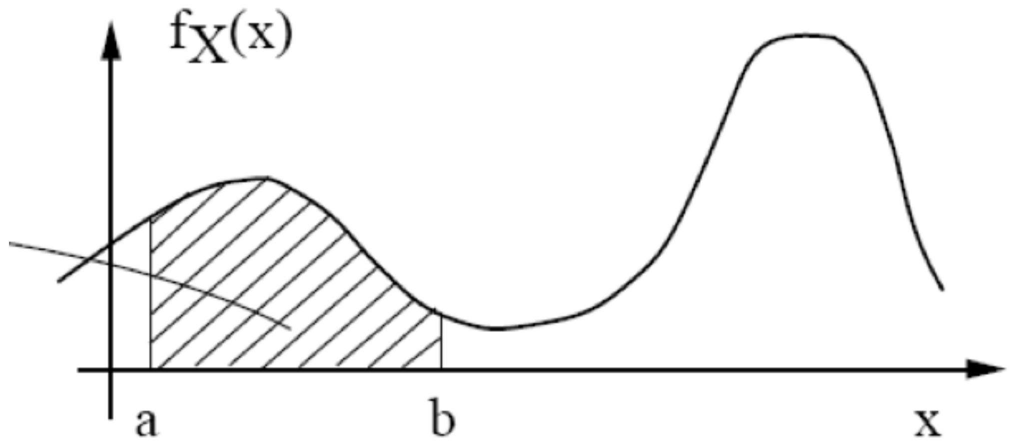
- Let X be a random variable whose value is drawn uniformly at random from [0,1]. What is $P(X = 0.5)$?

The Probability Density Function

In place of the PMF, we introduce the probability density function (or PDF), denoted f_X. For any a and any $b \geq a$,

$$P(a \leq X \leq b) = \int_a^b f_X(k)\,dk$$



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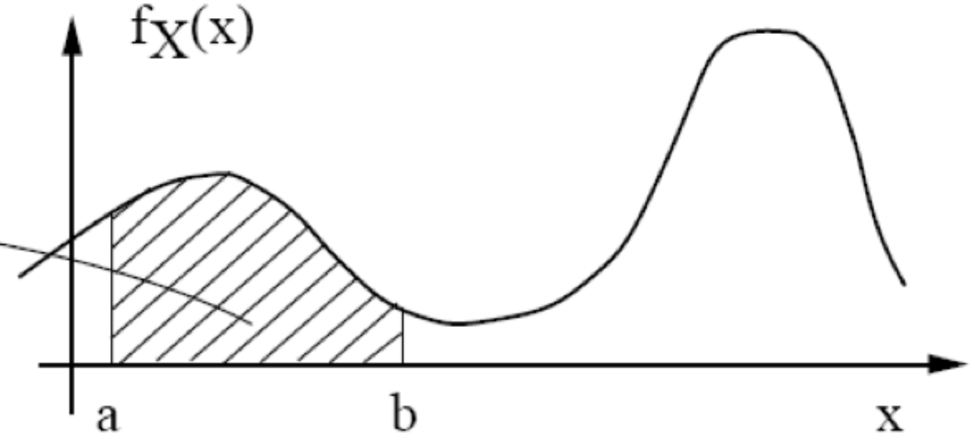

To satisfy normalization, we need $\int_{-\infty}^{\infty} f_X(k)\,dk = 1$

The Probability Density Function

For very small values of δ, we can approximate

$$P(a \leq X \leq a + \delta) = \int_{a}^{a+\delta} f_X(k)dk \approx f_X(a)\delta$$

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Need to be careful with this interpretation though!!! Note that, for example, $f_X(a)$ can be bigger than 1…

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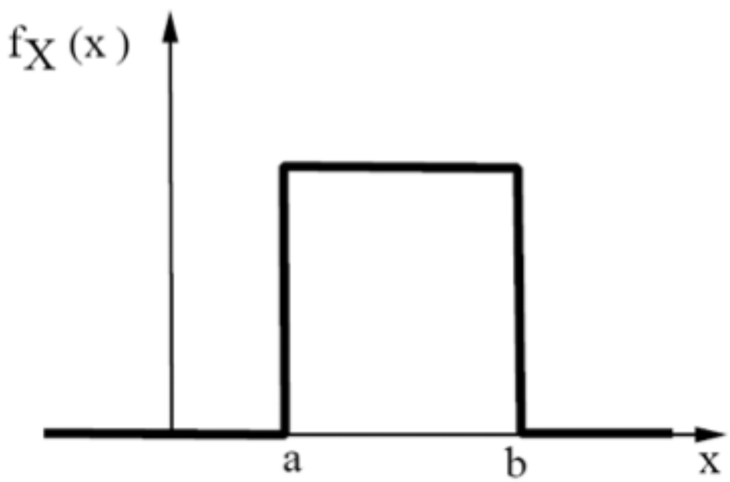
$$\mathrm{var}(X) = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (k - E[X])^2 f_X(k)\,dk$$

Uniform PDFs

Let X be a random variable whose value is drawn uniformly at random from the range [a, b]



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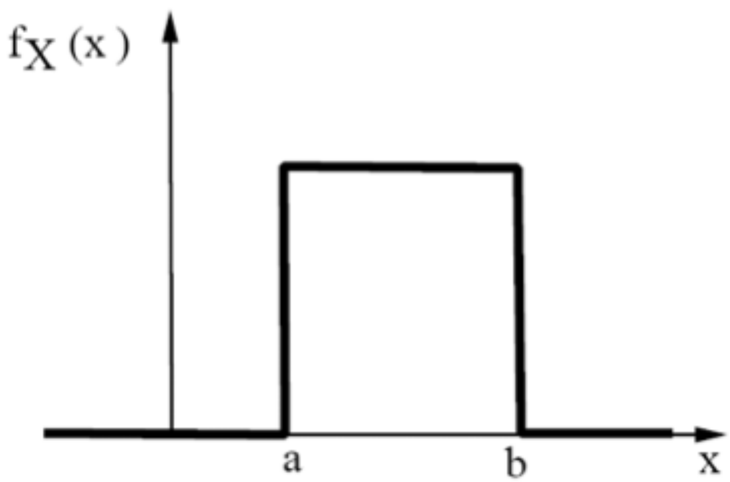

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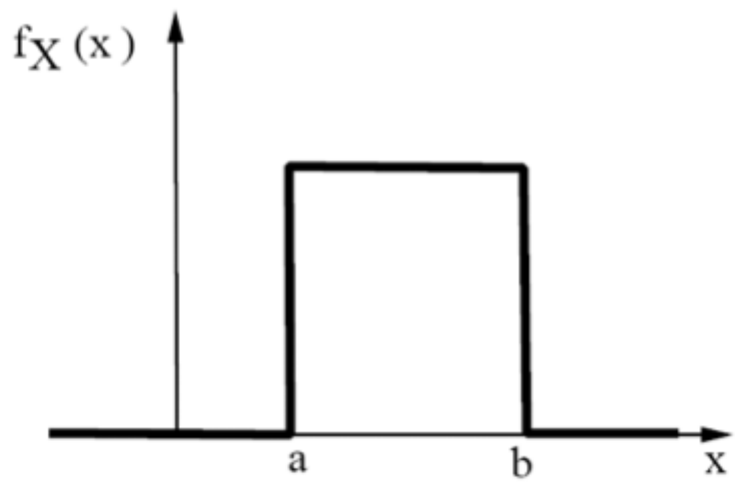

- What is f_X? What is E[X]? What is var(X)?
- What if X was chosen from the range (a, b) instead?

Exponential Random Variables

Exponential random variables model the amount of time until an incident of interest takes place

- Length of time before a message arrives at the computer
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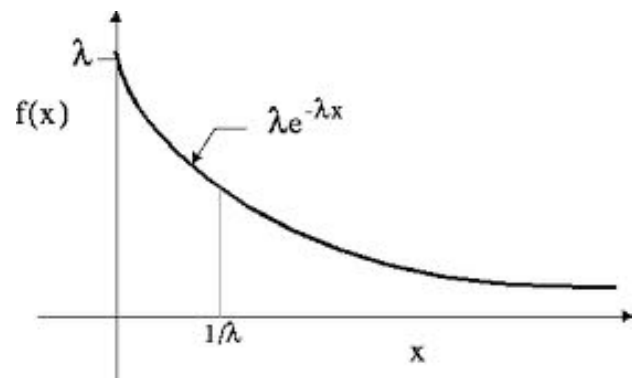

Figure 6. Exponential *pdf*

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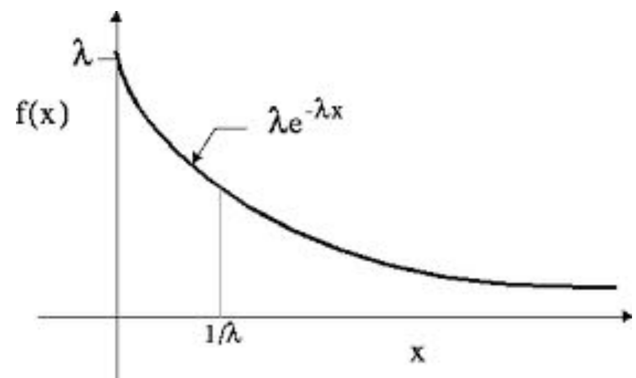

Figure 6. Exponential *pdf*

$E[X] = \lambda^{-1}$ $\quad$ $var(X) = \lambda^{-2}$

These can be verified using integration by parts.

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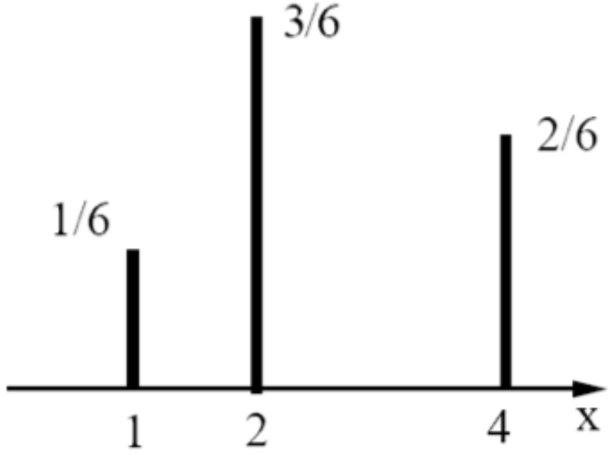
$$F_X(k) = \int_{-\infty}^{k} f_X(x)dx \quad \rightarrow \quad f_X(k) = \frac{dF_X}{dk}(k)$$

Cumulative Distribution Functions

Consider a discrete random variable X

PMF:



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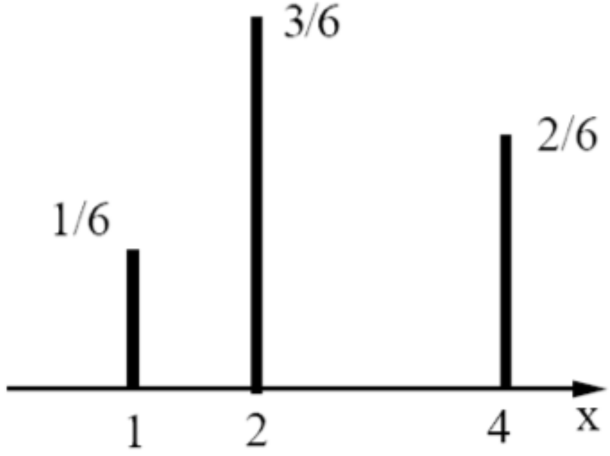
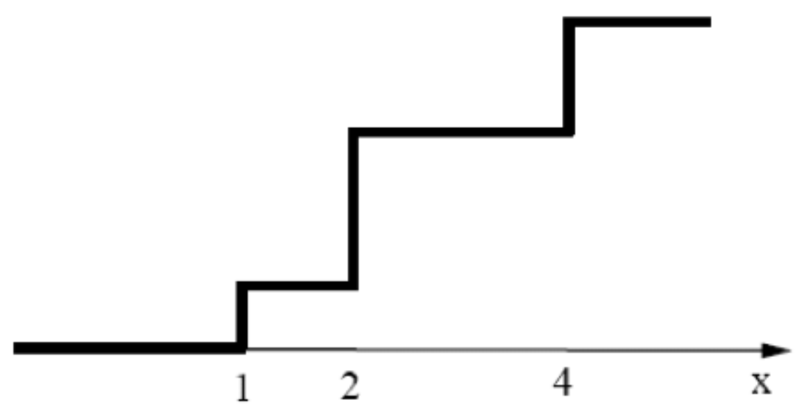
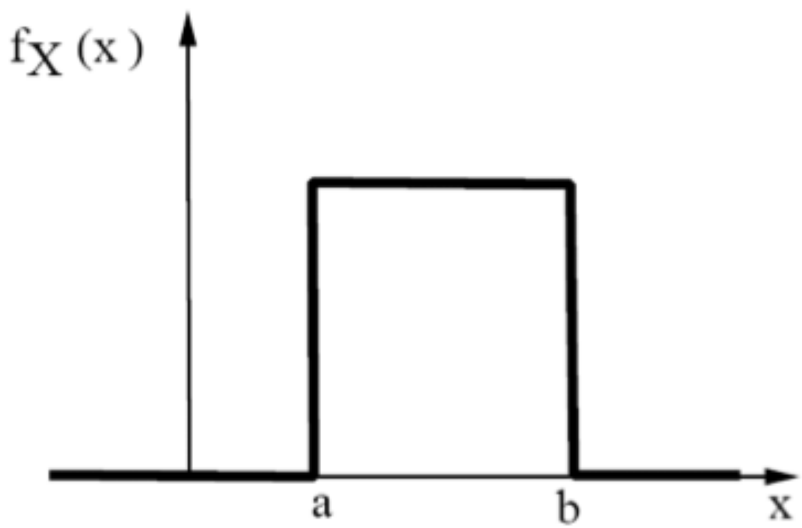


CDF:



Cumulative Distribution Functions

Consider a uniform continuous random variable X



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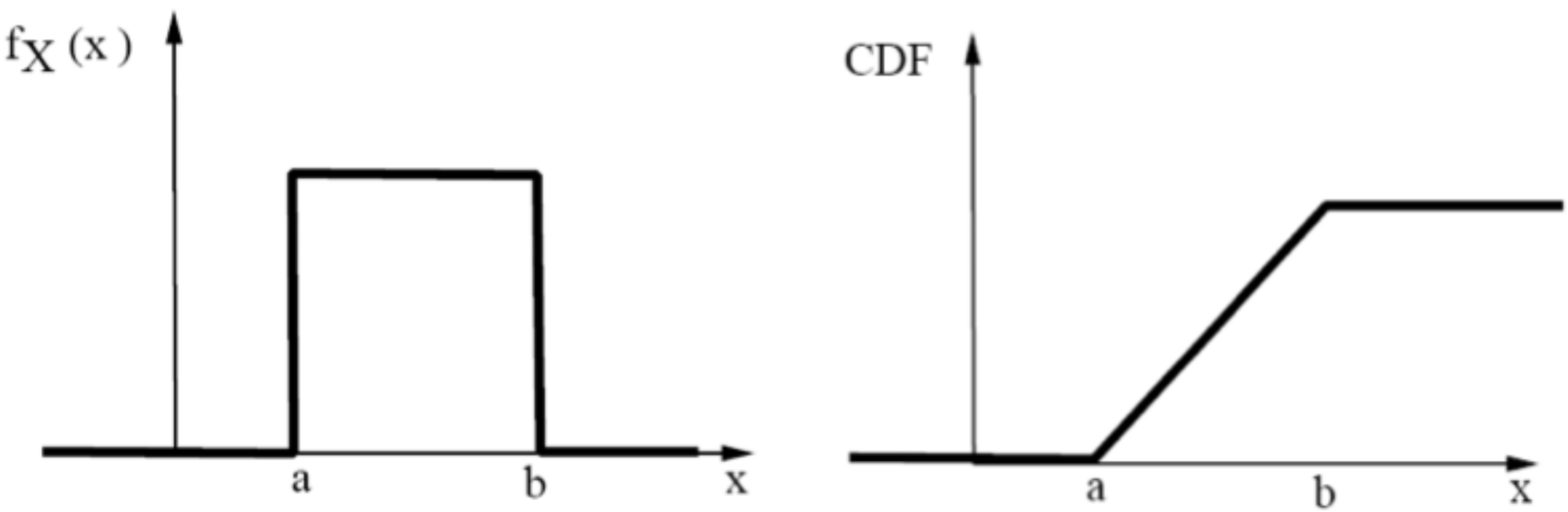

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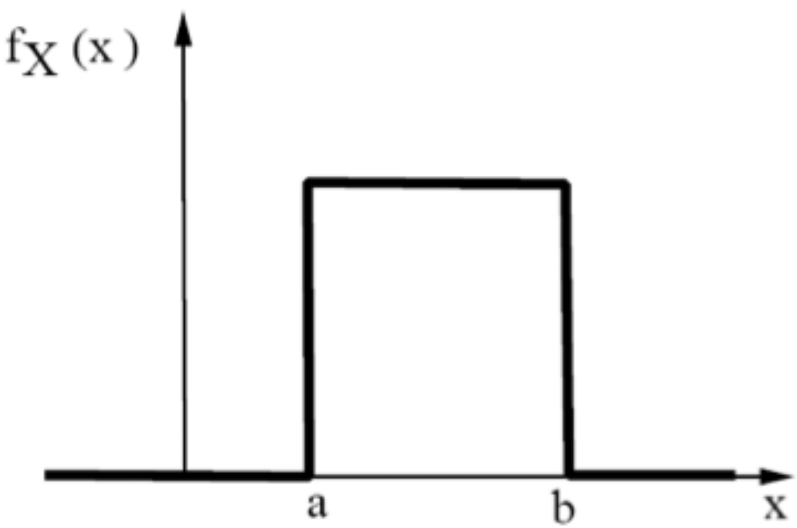

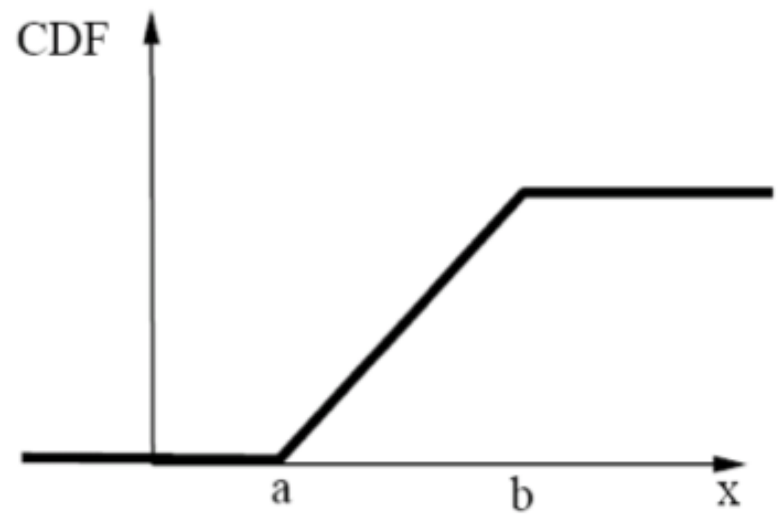

Note that CDFs are always monotonically non-decreasing

Cumulative Distribution Functions

What is the cumulative distribution function of an exponential random variable with $f_X(k) = \lambda e^{-\lambda k}$?

(we'll start with this next time...)