CS 112: Computer System Modeling Fundamentals

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Lecture 2

Reminders & Announcements

- Pick up a copy of the course textbook
- Take a look at the course website
  (especially the assigned reading!)
- Hand in your signed copy of the academic honesty policy
- Homework 1 has been posted!
  (due in class on April 12)

Last time…

- An experiment is an activity (real or conceptual) that results in exactly one of several possible outcomes.
- The set of all possible outcomes of an experiment is called the sample space, typically denoted Ω.
- An event is a collection of outcomes – that is, a subset of the sample space.
- The atomic events are the basic, fundamental elements of the sample space.

Last time…

Probability Axioms:

- Nonnegativity: $P(A) \geq 0$ for every event A
- Additivity: If A and B are disjoint events, then

$$P(A \cup B) = P(A) + P(B)$$

- Normalization: $P(\Omega) = 1$

Conditional Probability:

- When all outcomes are equally likely,

$$P(A \mid B) = \# \text{ elements in } A \cap B \; / \; \# \text{ elements in } B$$

- In general, $P(A \mid B) = P(A \cap B) / P(B)$

Today

- Examples of conditional probability
- Independence and conditional independence
- The Total Probability Theorem
- Bayes' Rule

Conditional Probability

Suppose we flip two fair coins.

- What is the probability that the first coin lands on tails given that the second coin lands on tails?

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Suppose I purchase a cookie with probability 0.5, a brownie with probability 0.25, and an apple with probability 0.25. What is the probability I purchase a cookie given that I purchase a baked good?

The Multiplication Rule

We can express $P(A_1 \cap A_2)$ in terms of a conditional probability:

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2 \mid A_1)$$

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More generally,

$$P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
$$= P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \ldots P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Independence

Let's go back to the two-coin example again…

- What is the probability that the first coin lands on tails given that the second coin lands on tails?

Independence

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What is the probability that the first coin lands on tails given that the second coin lands on tails?

Knowing whether or not the first coin lands on tails gives us no new information about how likely it is that the second coin lands on tails – these events are independent

Independence

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To prove or disprove independence, it is enough to prove or disprove any one of these conditions.

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- Consider events A and B such that $B \subset A$. Are A and B independent?

- Suppose we flip two fair coins. Is the event that the first flip is heads independent from the event that exactly one of the two flips is heads?

Conditional Independence

- Events A and B are conditionally independent given C if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

- Or equivalently (if all probabilities are nonzero), if

$$P(A \mid B \cap C) = P(A \mid C)$$

Conditional Independence

Back to the two-coin example again…

$T_1 = \{TH, TT\}$ "1st toss is a tail"

$T_2 = \{HT, TT\}$ "2nd toss is a tail"

$D = \{HT, TH\}$ "the tosses have different results"

We already know that T_1 and T_2 are independent.

Are they conditionally independent given D?

Uncertainty in LA

Every day I take the freeway to work with probability 1/3; otherwise I take back roads. When I take the freeway, I get stuck in traffic with probability 9/10. When I take the back roads, I get stuck in traffic with probability 1/4.

What is the probability that I get stuck in traffic?

Total Probability Theorem

Let A_1, …, A_n be pairwise disjoint events with nonzero probability such that $P(A_1 \cup \ldots \cup A_n) = 1$. Then for any event B,

$$P(B) = \sum_{i=1}^{n} P(A_i) P(B \mid A_i)$$

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Exercise: Try to prove this at home using the definition of conditional probability and the additivity axiom

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What is the probability that I get stuck in traffic?

Given that I got stuck in traffic, what is the probability that I took the freeway?

Bayes' Rule

Let A_1, …, A_n be pairwise disjoint events with nonzero probability such that $P(A_1 \cup \ldots \cup A_n) = 1$. Then for any event B with nonzero probability, for any *i*,

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)}$$

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First part holds for any two events as long as P(B) > 0

$$= \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}$$

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Given that I got stuck in traffic, what is the probability that I took the freeway?

Stress Test

A new stress test can predict whether or not a student is stressed with 80% accuracy; when a student is stressed, it correctly outputs positive with probability 0.8, while if a student is not stressed, it correctly outputs negative with probability 0.8. 95% of students are stressed.

If a student's test results are negative, what is the probability that the student is stressed?