CS 112: Computer System Modeling Fundamentals

Prof. Jenn Wortman Vaughan

June 2, 2011

Lecture 19

Reminders & Announcements

- Tomorrow's section will be a review session for the final
- Homework 5 is due at the start of section
- I will be traveling next week and will not have office hours
- Ethan's office hours for next week:
  - Monday, 11:30 – 12:30
  - Wednesday, 11:30 – 12:30

Final Exam

- Thursday, June 9, 3 – 6pm, in this room
- One double-sided sheet of hand-written notes allowed
- No other notes, books, calculators, cell phones, etc.
- Best way to study is to practice solving problems like the end-of-chapter problems in the book (solutions available on the book's website)

You should be able to…

Figure out when and how to apply the basic rules and definitions of probability, including

- the multiplication rule
- Bayes' rule
- the total probability rule
- the total expectation rule
- the definition of conditional probability
- the definition of independence
- the counting principle
- the definitions of expectation and variance

You should be able to…

- Translate word problems into math
  - Define the appropriate random variables or events
  - Be able to state the quantity that you must solve for in terms of these random variables or events
  - Recognize common random variables (such as Bernoulli, binomial, and geometric) in problem descriptions

Topics Covered

- Sample spaces, events, and basic probability (Chapter 1)
- Discrete and continuous random variables (Chapters 2–3)
- Covariance and correlation (4.2)
- Markov, Chebyshev, law of large numbers (5.1, 5.2, 5.4)
- MAP and maximum likelihood (8.1, 8.2, parts of 9.1)
- Naive Bayes (see link on website)
- Bayesian networks (see link on website)
- Markov chains (7.1–7.4)

You are responsible for anything covered in the book or class!

Convergence of Markov Chains

Theorem: Consider any Markov chain with a single recurrent class, which is not periodic. Then the steady state probabilities of the chain are the unique values $\pi_1, \ldots, \pi_m$ that satisfy the system of equations:

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{k,j} \qquad \text{for } j = 1, \ldots, m$$

$$\sum_{k=1}^{m} \pi_k = 1$$

These are referred to as the balance equations.

When these conditions hold, X_0 doesn't matter.

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- What if there are multiple recurrent states?
  - Once a recurrent state is entered, the same ideas apply
  - How likely are we to enter each recurrent state?

The Gambler's Ruin

A gambler plays rounds of games in a casino. At each round, he wins \$1 with probability p and loses \$1 with probability 1-p, independent of what happens in other rounds. He continues playing until he either accumulates a target amount of money \$$m$, or loses all of his money.

- How can we represent this scenario as a Markov chain?

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- How can we represent this scenario as a Markov chain?
- What are the recurrent classes?
- What is the probability that the gambler accumulates \$$m$?

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What do we know about the values a_i?

Absorption Probability Equations

Theorem: Consider a Markov chain in which each state is either absorbing or transient. Fix a particular absorbing state s. Then the probabilities a_i of eventually reaching state s after starting at state i are the unique solutions to the following system of equations:

$$a_s = 1$$

$$a_i = 0 \qquad \text{for all absorbing } i \neq s$$

$$a_i = \sum_{j=1}^{m} p_{i,j} a_j \qquad \text{for all transient } i$$

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- By merging states, we are able to apply the same ideas…

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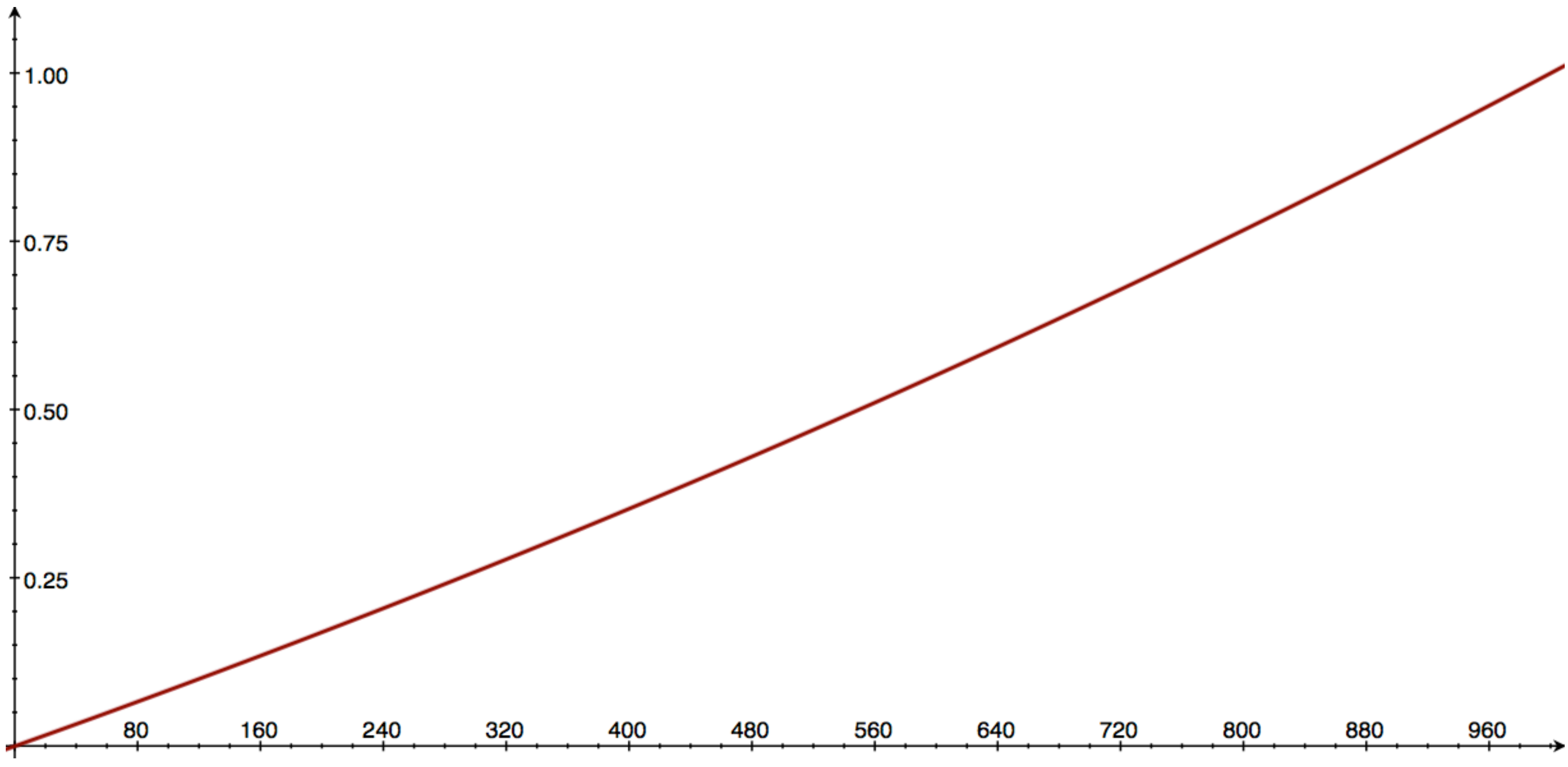
- What is the probability that the gambler accumulates \$$m$?

(we'll just do the case where p = .5, but you can read the general solution in example 7.11 in the text)

Gambler's Ruin

$m = 1000 \qquad p = 0.4999$



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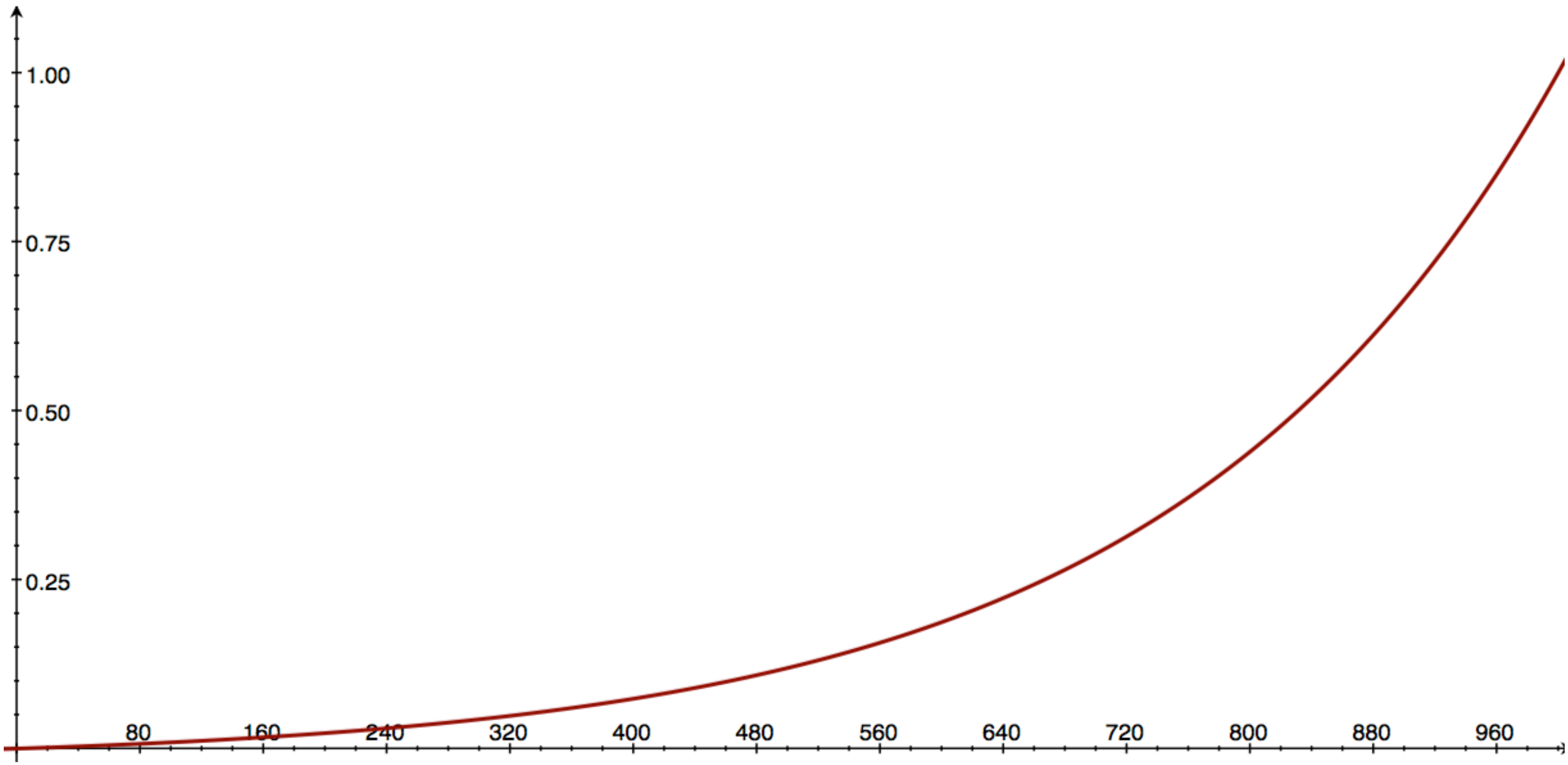

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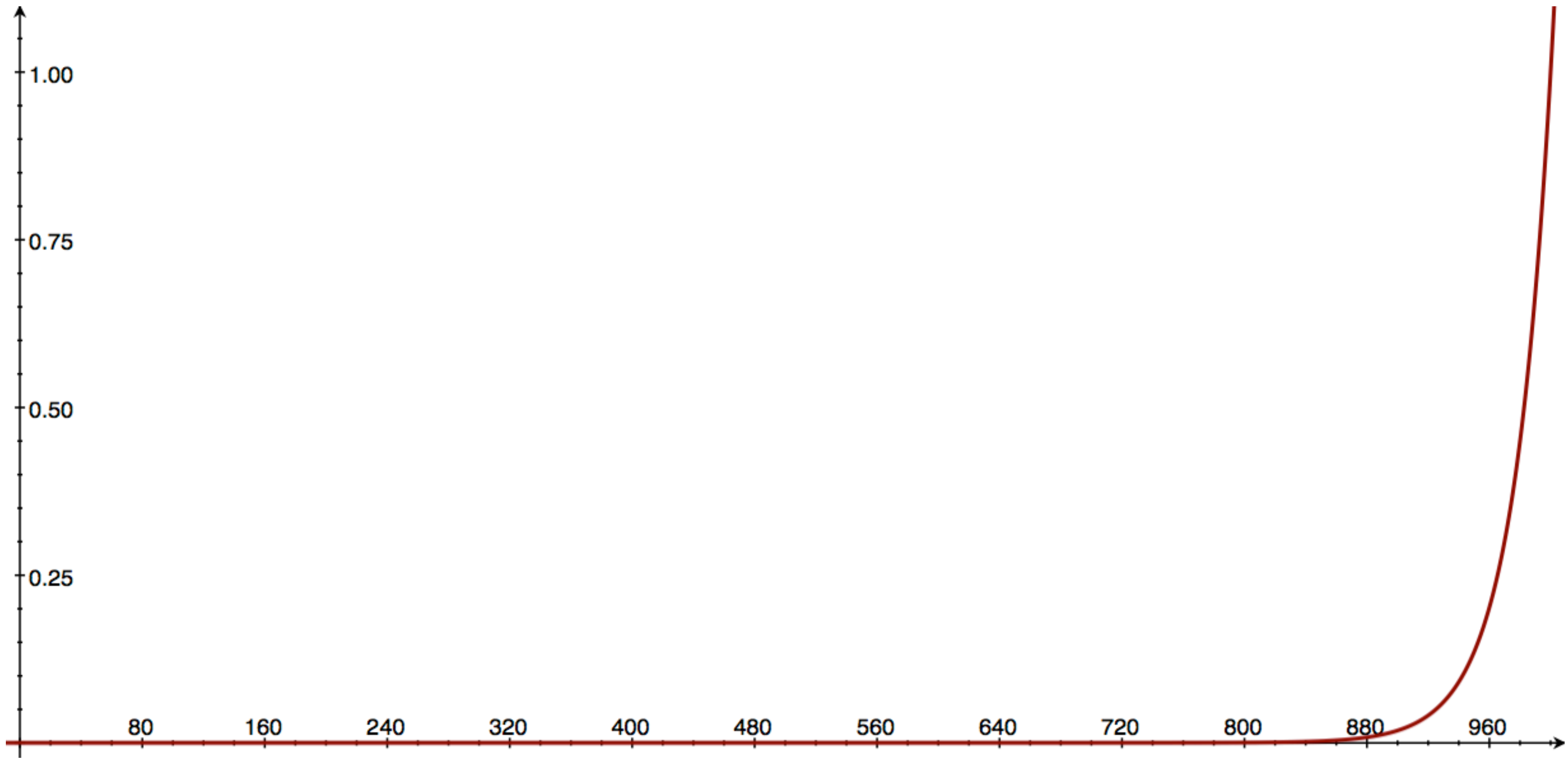
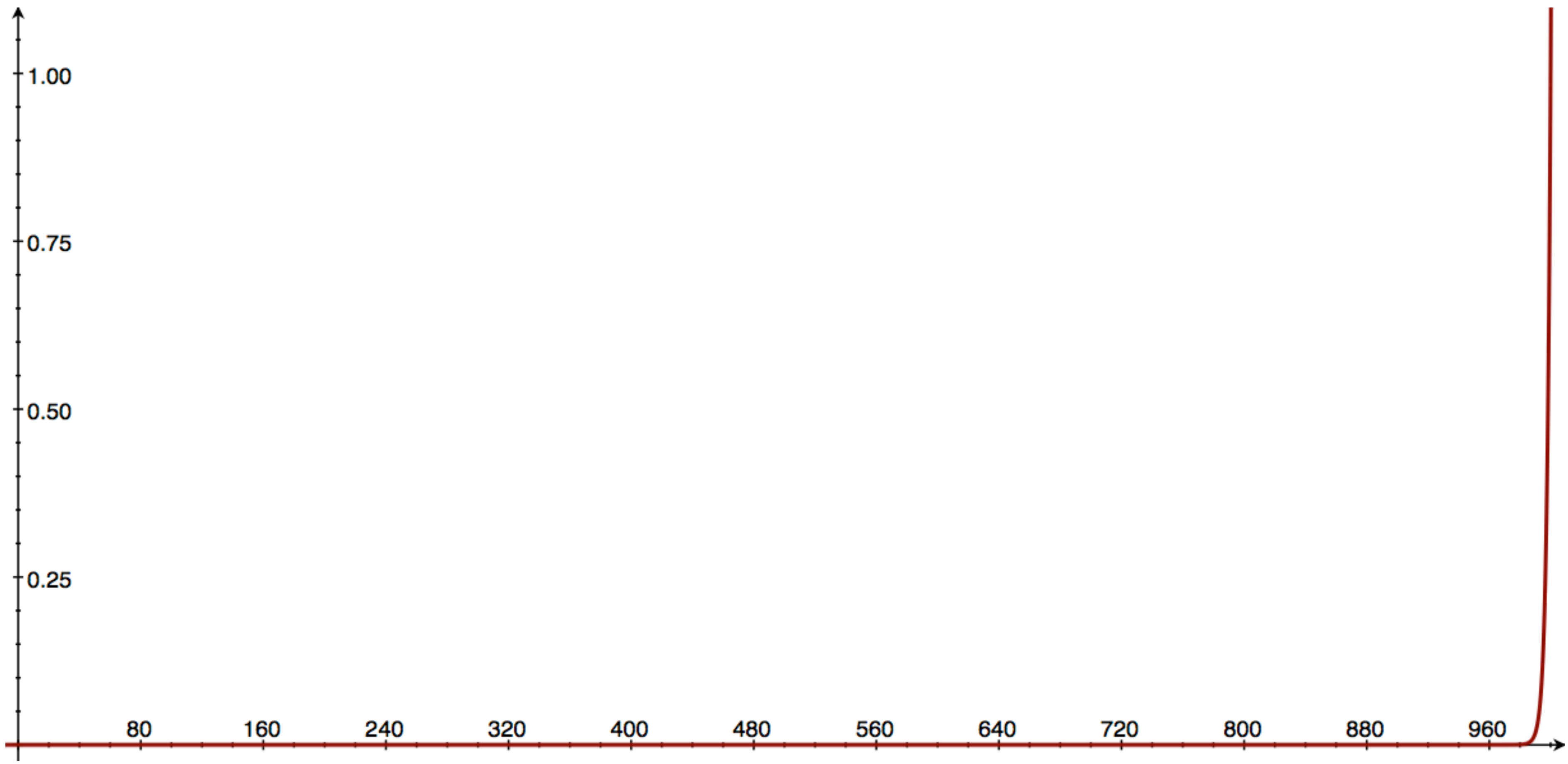


Gambler's Ruin

$m = 1000 \quad p = 0.4$



Good luck on the final
& enjoy the summer!