CS 112: Computer System Modeling Fundamentals

Prof. Jenn Wortman Vaughan

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Lecture 17

Reminders & Announcements

- Homework 5 has been posted on the website
- Catch up on your reading now!

Reasoning About Complex Scenarios

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  - The sequence of daily prices of a stock
  - The sequence of websites a web surfer visits (PageRank)
  - The number of packets in a network buffer over time
  - The sequence of words in an “English-looking” document

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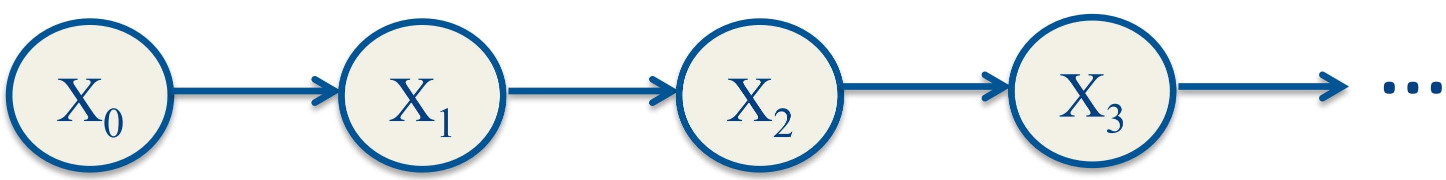
The key independence assumption is the Markov property, which we can represent using a Bayesian network:



A Simple Example

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

- What are the states?
- What are the transition probabilities?

Representing Transition Probabilities

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- The best representation depends on the problem you are trying to solve

The Markov Property

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

If it remains offline for four straight days, I get it (temporarily) repaired, resetting it to the online state.

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$$P(X_n = j \mid X_0 = i) = \sum_{k=1}^{m} p_{k,j} P(X_{n-1} = k \mid X_0 = i)$$

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Exercise: Derive this recursion yourself using the total probably rule and the probability of a sequence of states!

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 1) = 1$ $P(X_0 = 2 \mid X_0 = 1) = 0$

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$P(X_0 = 1 \mid X_0 = 1) = 1$ $\qquad$ $P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 1 \mid X_0 = 1) = 0.8$ $\qquad$ $P(X_1 = 2 \mid X_0 = 1) = 0.2$

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$P(X_2 = 1 \mid X_0 = 1) = 0.76$ $\qquad$ $P(X_2 = 2 \mid X_0 = 1) = 0.24$

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$P(X_1 = 1 \mid X_0 = 1) = 0.8$

$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_3 = 1 \mid X_0 = 1) = 0.752$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

$P(X_3 = 2 \mid X_0 = 1) = 0.248$

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$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_3 = 1 \mid X_0 = 1) = 0.752$

$P(X_4 = 1 \mid X_0 = 1) = 0.7504$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

$P(X_3 = 2 \mid X_0 = 1) = 0.248$

$P(X_4 = 2 \mid X_0 = 1) = 0.2496$

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$P(X_1 = 1 \mid X_0 = 1) = 0.8$

$P(X_2 = 1 \mid X_0 = 1) = 0.76$

$P(X_3 = 1 \mid X_0 = 1) = 0.752$

$P(X_4 = 1 \mid X_0 = 1) = 0.7504$

$P(X_5 = 1 \mid X_0 = 1) = 0.7501$

$P(X_0 = 2 \mid X_0 = 1) = 0$

$P(X_1 = 2 \mid X_0 = 1) = 0.2$

$P(X_2 = 2 \mid X_0 = 1) = 0.24$

$P(X_3 = 2 \mid X_0 = 1) = 0.248$

$P(X_4 = 2 \mid X_0 = 1) = 0.2496$

$P(X_5 = 2 \mid X_0 = 1) = 0.2499$

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My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$ $P(X_0 = 2 \mid X_0 = 2) = 1$

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My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$ $\qquad$ $P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$ $\qquad$ $P(X_1 = 2 \mid X_0 = 2) = 0.4$

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My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

$P(X_0 = 1 \mid X_0 = 2) = 0$

$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_3 = 1 \mid X_0 = 2) = 0.744$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

$P(X_3 = 2 \mid X_0 = 2) = 0.256$

n-Step Transitions

My faulty router can be either online or offline. If it is online one day, it will be online the next day with probability 0.8. If it is offline one day, it will remain offline the next day with probability 0.4.

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$P(X_1 = 1 \mid X_0 = 2) = 0.6$

$P(X_2 = 1 \mid X_0 = 2) = 0.72$

$P(X_3 = 1 \mid X_0 = 2) = 0.744$

$P(X_4 = 1 \mid X_0 = 2) = 0.7488$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

$P(X_3 = 2 \mid X_0 = 2) = 0.256$

$P(X_4 = 2 \mid X_0 = 2) = 0.2512$

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$P(X_3 = 1 \mid X_0 = 2) = 0.744$

$P(X_4 = 1 \mid X_0 = 2) = 0.7488$

$P(X_5 = 1 \mid X_0 = 2) = 0.7498$

$P(X_0 = 2 \mid X_0 = 2) = 1$

$P(X_1 = 2 \mid X_0 = 2) = 0.4$

$P(X_2 = 2 \mid X_0 = 2) = 0.28$

$P(X_3 = 2 \mid X_0 = 2) = 0.256$

$P(X_4 = 2 \mid X_0 = 2) = 0.2512$

$P(X_5 = 2 \mid X_0 = 2) = 0.2502$

Classification of States

Accessibility: State j is accessible from state i if for some n, the n-step transition probability from i to j is positive.

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If *i* is not recurrent, then it is transient.

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If *i* is not recurrent, then it is transient.

The set of all states accessible from a recurrent state form a recurrent class. Note that all of the states in a recurrent class are accessible from each other.

Periodicity

- A recurrent class is periodic if it can be broken into $d > 1$ disjoint subsets $S_1, S_2, \ldots, S_d$ in such a way that
  - All transitions from states in S_i lead to states in S_{i+1} for $i \in \{1, \ldots, d-1\}$
  - All transitions from states in S_d lead to states in S_1
- The period of the class is the number d of subsets