CS112: Computer System Modeling Fundamentals
Homework 1

Due Tuesday, April 12, 4pm (in class)

Please refer to the course academic integrity policy for collaboration rules, and be sure to make use of the problem solving guidelines covered in section. Start early!

1. There are two events A and B, both with nonzero probabilities. If the occurrence of B makes the occurrence of A more likely (that is, if $\mathbf{P}(A|B) > \mathbf{P}(A)$), is it ALWAYS true that the occurrence of A also makes the occurrence of B more likely? Provide a proof or counterexample to support your answer. (A proof should be rigorous, and a counterexample should include well-specified probabilities for any events of interest.)

2. We are told that events A and B are independent. In addition, events A and C are independent. Is it ALWAYS true that A is independent of $B \cup C$? Provide a proof or counterexample.

3. The disc containing your favorite MP3 song on it has been scratched very badly. The disc was mixed up with three other scratched discs that were lying around. It is equally likely that any of the four discs holds your favorite song. A DJ friend of yours offers to take a look since he claims to recover songs from any disc with a 90% chance (assuming the song is there).

   (a) Given that he searches on disc 1 but cannot recover your song, what is the probability that your song is on disc i for $i = 1, 2, 3, 4$?

   (b) Now suppose your DJ friend has searched both disc 1 and disc 2 and has not yet found the file. What is the probability it is on disc i for $i = 1, 2, 3, 4$? (*HINT:* Given that the song is on a particular disc i, the event that the DJ finds it on disc 1 is independent of the event that the DJ finds it on disc 2.)

4. A parking lot consists of a single row containing n parking spaces for some reasonably large n (say, $n \geq 5$). Mary arrives when all spaces are free. Tom is the next person to arrive. Each person makes an equally likely choice among all available spaces at the time of arrival.

   (a) Describe a natural sample space for this experiment mathematically. (Remember that a sample space is defined as a *set* of possible outcomes.)

   (b) Obtain $\mathbf{P}(A)$, the probability the parking spaces selected by Mary and Tom are at most 2 spaces apart (that is, at most 1 empty space between them).

5. A company has a data center with a total of 1,000 computers. The computers are grouped into clusters of 25 computers each. A cluster fails if 2 or more of its computers fail.

   Given that 3 of the 1,000 computers fail on particular day, what is the probability that at least 1 cluster fails? Assume that all computers are equally likely to have failed.

6. Let A be an event such that $0 < \mathbf{P}(A) < 1$. The *odds in favor* of A are defined to be:

$$\boldsymbol{O}(A) = \frac{\mathbf{P}(A)}{\mathbf{P}(A^c)}$$

while the *odds against* A are defined to be the reciprocal of $\boldsymbol{O}(A)$.[1] This problem deals with a formula for calculating "conditional odds," that is, odds based on some partial information. If A and B are events with $\mathbf{P}(A) > 0$ and $\mathbf{P}(B) > 0$, the odds in favor of A given B are dened as

$$\boldsymbol{O}(A|B) = \frac{\mathbf{P}(A|B)}{\mathbf{P}(A^c|B)}.$$

Show that

$$\boldsymbol{O}(A|B) = \boldsymbol{L}(B|A)\boldsymbol{O}(A),$$

where $\boldsymbol{L}(B|A)$ is the so-called *likelihood ratio* of B given A, defined as

$$\boldsymbol{L}(B|A) = \frac{\mathbf{P}(B|A)}{\mathbf{P}(B|A^c)} \ .$$

7. May B. Lucky is a compulsive gambler who is convinced that on any given day she is either "lucky," in which case she wins each red/black bet she makes in the roulette with probability $p_L > 1/2$, or she is "unlucky," in which case she wins each red/black bet she makes in the roulette with probability $p_U < 1/2$. May visits the casino every day, and believes that she knows the a priori probability that any one given visit is a "lucky" one (i.e., corresponds to p_L rather than p_U). To improve her chances, May adopts a system whereby she estimates on-line whether she is lucky or unlucky on a given day, by keeping a running count of the numbers of bets that she wins and loses. In particular, she continues to play until the conditional odds in favor of the event lucky on the current day, given the number of wins and losses so far, fall below a certain threshold (see the preceding problem). As soon as this happens, she stops playing.

   Provide a simple algorithm for updating May's conditional odds with each play.

[1] To connect the term "odds" with its common usage, note for example that if the probability that a given horse wins a race at the track is 1/3, the odds against the horse winning are 2 to 1. A "fair" racetrack would then pay \$2 for every \$1 bet on the horse (plus the original \$1 bet) if the horse wins.